Berkeley Math Circle
Monthly Contest 2
Due November 4, 2014

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 1, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Six consecutive prime numbers have sum p. Given that p is also prime, determine all possible values of p.

2. With two properly chosen weights and a balance scale, it is possible to determine the weight of an unknown object known to weigh an integer number of pounds from 1 to n. Find the largest possible value of n.

   Remark. The balance scale tells whether the weights placed on each side are equal and, if not, which side is heavier. It may be used an unlimited number of times.

3. Let ABC be a triangle. A circle is tangent to segments BC, CA, AB at points D, E, F, respectively. Given that the measures of $\angle CAB$, $\angle ABC$, $\angle BCA$ form an arithmetic progression in some order, prove that the measures of $\angle FDE$, $\angle DEF$, $\angle EFD$ also form an arithmetic progression in some order.

4. On a distant planet, there are 2014 cities, some pairs of which are connected by two-way roads. It turns out that the population of each city is the average of the populations of the cities to which it is connected by a single road, and moreover that it is possible to travel from every city to every other city by a sequence of roads.

   Prove that all cities have the same population.

5. Prove that, for positive integers n and m,
$$\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1.$$

6. Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative real numbers. Find all functions $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $x, y \in \mathbb{R}_{\geq 0}$,
$$f\left(\frac{x + f(x)}{2} + y\right) = 2x - f(x) + f(f(y))$$
   and
$$(f(x) - f(y))(x - y) \geq 0.$$

7. Let ABC be a scalene triangle inscribed in circle Γ. The internal bisector of $\angle A$ meets $\overline{BC}$ and circle Γ at points D and E. The circle with diameter $\overline{DE}$ meets Γ at a second point F. Prove that $\frac{AB}{AC} = \frac{FB}{FC}$.