Berkeley Math Circle
Monthly Contest 1, Solutions

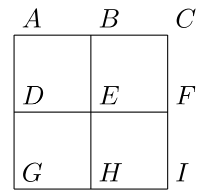
1. In the sequence

$$1, 4, 7, 10, 13, 16, 19, \ldots,$$

each term is 3 less than the next term. Find the 1000th term of the sequence.

Solution. To get from the first term to the 1000th term, we must add three 999 times. That amounts to adding $3 \cdot 999$ which is 2997. So the 1000th term is $1 + 2997 = 2998$.

2. Is it possible to draw the figure at right, without lifting your pencil from the sheet of paper,



(a) such that each line segment is drawn only once?

(b) such that each line segment is drawn exactly twice?

Tricks such as folding the paper, drawing on both sides, or retracting a mechanical pencil are not permitted.

Solution. (a) Consider the point marked B in the accompanying diagram. It has three line segments coming out of it. If the path does not start at B, it will eventually have to enter B by one of these line segments and then leave it by another. Then at some point it must traverse the third of these line segments and arrive at B again, where there is nowhere to go. In summary, B must be either the start or the end of the path. In the same way we can show that D, F, and H must also be one of the two endpoints of the path. But there are only two endpoints so we have a contradiction.

(b) It is possible. Here is one of an extremely large number of solutions:

$$ABCFEDGHIFIHEHGDADEBEFCBA.$$

It is easy to check that each of the twelve unit line segments is used exactly twice. (Note also that the path starts and ends at the same point; the reader is invited to prove that this is necessarily so.)

3. Six children are invited to a birthday party, and each pair of them are either mutual friends or mutual strangers. Prove that there are either three of them that are all friends or three of them that are all strangers to one another.

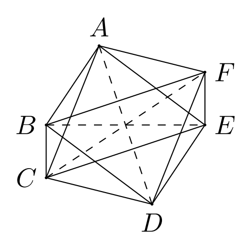
Solution. Begin by letting A be any person at the party, and note that A must be either friends or strangers with at least three of the others, for otherwise there would only be at most $2 + 2 = 4$ other people at the party. Because of the symmetry between friends and strangers, we can assume that A has three friends, call them B, C, and D. Now if any two of these three are friends, then they together with A are the desired triple. Otherwise, B, C, and D are all mutual strangers, so they themselves furnish the desired triple.

4. Determine all integers x such that the product $x(x + 1)(x + 2)$ is the square of an integer.

Solution. The answers are $x = 0$, $x = -1$, and $x = -2$, for all of which the product is $0 = 0^2$.

Assume that there is another solution. We can immediately rule out the case $x < -2$, as then the product is negative. So x is a positive integer. Note that the greatest common divisor of x and $x + 1$ is 1, as two consecutive integers cannot be multiples of the same prime. Likewise, $\gcd(x + 2, x + 1) = 1$ and so $\gcd(x(x + 2), x + 1) = 1$. Now $x(x + 2)$ and $x + 1$ are relatively prime positive integers whose product is a square, so they must both be squares (any prime dividing either of them must do so to an even power). But $x(x + 2) = x^2 + 2x = (x + 1)^2 - 1$, so now $x(x + 2)$ and $(x + 1)^2$ are two consecutive positive integers which are both squares, which is impossible (the difference between any two consecutive squares a^2 and $(a + 1)^2$ is $2a + 1 \geq 3$).

5. Let $ABCDEF$ be a convex hexagon such that the quadrilaterals $ABDE$ and $ACDF$ are parallelograms. Prove that $BCEF$ is also a parallelogram.



Proof. It is well known that a quadrilateral is a parallelogram if and only if the two diagonals bisect each other, that is, have a common midpoint. Since $ABDE$ is a parallelogram, the midpoints of AD and BE coincide; since $ACDF$ is a parallelogram, the midpoints of AD and CF coincide. Consequently the midpoints of BE and CF coincide and $BCEF$ is a parallelogram. □

6. Determine if it is possible to color each of the rational numbers either red or blue such that the following three conditions are all satisfied:

(i) x and $-x$ are opposite colors, for all rational $x \neq 0$;

(ii) x and $1-x$ are opposite colors, for all rational $x \neq 1/2$;

(iii) x and $1/x$ are opposite colors, for all rational $x \neq 0, \pm 1$.

Solution. The answer is yes.

We will prove the following statement by induction: It is possible to color the rational numbers with denominator at most n red and blue such that the conditions (i)–(iii) hold whenever the two rational numbers in question both have denominator at most n, and such that no number reverses color at any step. (The last condition ensures that, when the induction is complete, every number will have a well-defined color.)

We make two base cases. For $n = 1$, color all the positive integers red and all the nonpositive integers blue. Then, for $n = 2$, color the number $x + \frac{1}{2}$ (for integer x) blue if $x \geq 0$ and red if $x < 0$. It is easy to see that conditions (i)–(iii) are satisfied so far.

Assume that all rational numbers with denominator less than n ($n \geq 3$) have been colored. We must color the fractions m/n, where m is an integer coprime to n. First, when $0 < m < n$, color m/n the opposite of n/m (which has already been colored). Then color $1 + m/n$ the same color as m/n for each $m > 0$ coprime to n. Finally, color $-m/n$ the opposite of m/n. Condition (i) is obviously satisfied. Condition (iii), that m/n and n/m are opposite colors, is relevant only if $|m| < n$; it is satisfied by definition if $m > 0$, and otherwise it is enough to note that the row of numbers

$$\frac{m}{n}, \quad \frac{n}{m}, \quad -\frac{n}{m}, \quad -\frac{m}{n}$$

has been colored alternately red and blue, implying that the two end terms have opposite colors. It remains to verify condition (ii). If $x > 1$ or $x < 0$, this follows from (i) and the fact that y and $y + 1$ have been colored alike for $y > 0$. So we are left with the case $0 < x < 1$, that is, proving that m/n and $(n-m)/n$ are the same color when $0 < m < n$. The six numbers

$$\frac{m}{n}, \frac{n}{m}, -\frac{n-m}{m}, -\frac{m}{n-m}, \frac{n}{n-m}, \frac{n-m}{n}$$

have been colored alternately red and blue by (ii) and (iii), so the two end terms have opposite colors.

Remark. An explicit description of the coloring can be given in terms of continued fractions: every rational number $x > 0$ has a unique expression

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

where the a_i are nonnegative integers, positive except for a_0, and $a_n \geq 2$ (if $n \geq 1$). The above inductive procedure colors x red if n is even and blue if n is odd.

7. Let $a > 2$ be given, and define a sequence $a_0, a_1, a_2, \ldots$ by

$$a_0 = 1, \quad a_1 = a, \quad a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2\right) \cdot a_n.$$

Show that for all integers $k \geq 0$, we have

$$\sum_{i=0}^{k} \frac{1}{a_i} < \frac{a + 2 - \sqrt{a^2 - 4}}{2}.$$

Solution. Write the recursion as

$$\frac{a_{n+1}}{a_n} = \left(\frac{a_n}{a_{n-1}}\right)^2 - 2.$$

Let $b_n = a_{n+1}/a_n$; then $b_n = b_{n-1}^2 - 2$ and $b_0 = a$. Let

$$t = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Then $a = t + 1/t$ and it is a simple induction to show that $b_n = t^{2^n} + t^{-2^n}$ for $n \geq 0$. Then

$$a_n = a_0 b_0 b_1 \cdots b_{n-1} = \prod_{i=0}^{n-1} \left(t^{2^i} + t^{-2^i} \right) = \prod_{i=0}^{n-1} \frac{t^{2^{i+1}} - t^{-2^{i+1}}}{t^{2^i} - t^{-2^i}} = \frac{t^{2^n} - t^{-2^n}}{t - t^{-1}}.$$

So

$$\begin{aligned}
\sum_{i=0}^{k} \frac{1}{a_i} &= (t - t^{-1}) \sum_{i=0}^{k} \frac{1}{t^{2^i} - t^{-2^i}} \\
&= (t - t^{-1}) \sum_{i=0}^{k} \frac{t^{2^i} + 1 - 1}{t^{2^{i+1}} - 1} \\
&= (t - t^{-1}) \sum_{i=0}^{k} \left(\frac{1}{t^{2^i} - 1} - \frac{1}{t^{2^{i+1}} - 1} \right) \\
&= (t - t^{-1}) \left(\frac{1}{t - 1} - \frac{1}{t^{2^{k+1}} - 1} \right) \\
&< (t - t^{-1}) \left(\frac{1}{t - 1} \right) \\
&= 1 + \frac{1}{t} \\
&= \frac{a + 2 - \sqrt{a^2 - 4}}{2}.
\end{aligned}$$