# TILINGS at Berkeley Math Circle!

Inspired by Activities of Julia Robinson Math Festival and Nina Cerutti and Leo B. of SFMC.

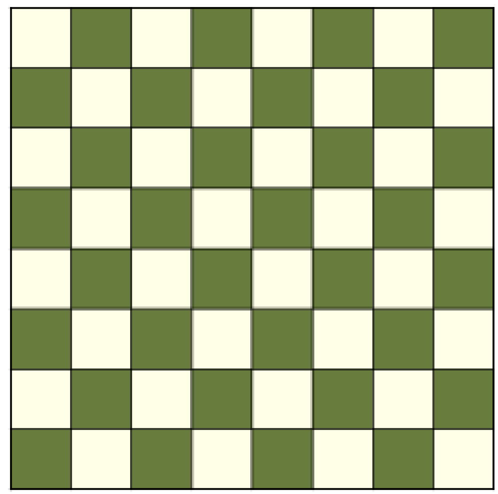
Julia Robinson Mathematics Festival

Directed By Joshua Zucker

## Tiling Torment

**The problem**

There are many problems that involve tiling (covering) all the squares on a chessboard (or similar board) with tiles of various sizes. The chessboard may be 8x8, 7x7 or other sizes and may or may not have squares missing. The tiles can be dominoes (2x1) or tiles of other sizes.



**Questions**

**The Basics**

1. Is it possible to tile a 7x7 board with 2x1 tiles?

2. In general, is it possible to tile an nxn board with 2x1 tiles? If so, which boards can you tile and why?

**Taking it Further**

3. Now consider the 7x7 board again. If you remove one square, is it possible to tile the board? If so, does it matter which square you remove? Describe completely.

4. In general, if n is odd, is it possible to tile an nxn board with 2x1 tiles if one square is covered with a 1x1 tile? Does it matter which square is covered?

5. Remove two diagonally opposite corners of a chessboard. Is it possible to tile this shape with 31 2x1 tiles?

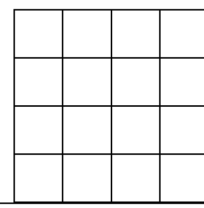
6. In general, if n is even, is it possible to tile an nxn board with 2x1 tiles if two squares are removed? Does it matter which two squares are removed?

7. Is it possible to tile an 8x8 board with 21 “L-shaped” tiles of three squares and one 1x1 tile? If so, how? Describe all possible locations for the 1x1 tile. If not, why not?

**SPECIAL Tiles - $2^i$ x $2^i$ Squares….**

Suppose you have an unlimited amount of Special Tiles that are formed with the following dimensions: The length of the sides of each of the Tiles is of the form $2^i$, where i is a nonnegative integer. *For example, when i = 2 we have a tile of size $2^2 x 2^2 = 4x4$.*

Using what you know about $2^i$ = 2 multiplied by itself i times, figure out the dimensions of other Tiles (add pictures if that's helpful for you):

| **i** | **$2^i$** | **Tile Size** (Square of $2^i$ x $2^i$) |
|---|---|---|
| i = 0 | | |
| i = 1 | | |
| i = 2 | $2^2 = 4$ | $2^2$ x $2^2$ = 4x4  |
| i = 3 | | |
| i = 4 | | |
| i =5 | | |
| i = 6 | | |

**1-Dimensional Tiling…**

When we tile something in 1-D using our special Tiles, we don't care about their height, we only care about using their width of $2^i$ to tile the length $2^i$ of the total line segment length.

1. Draw a line of length 23.
   a. What is the greatest amount of Tiles of size $2^i$ x $2^i$ can you use to "tile" this line?
   b. What is the least amount of squares you can use to tile this line?

2. Suppose that you are only allowed to use one of each Tile size.
Which of the following line segment can you tile under the given restrictions? If you find a way to tile the line, indicate the pattern by filling in the number 1 on the tiles that you used to tile that line segment length.

| | $2^4$x$2^4$ Tile | $2^3$ x $2^3$ Tile | $2^2$x$2^2$ Tile | $2^1$x$2^1$ Tile | $2^0$x$2^0$ Tile |
|---|---|---|---|---|---|
| **1** | | | | | |
| **2** | | | | | |
| **3** | | | | | |
| **4** | | | | | |
| **5** | | | | | |
| **6** | | | | | |
| **7** | | | | | |
| **8** | | | | | |
| **9** | | | | | |
| **10** | | | | | |
| **11** | | | | | |
| **12** | | | | | |
| **13** | | | | | |
| **14** | | | | | |
| **15** | | | | | |
| **16** | | | | | |
| **17** | | | | | |
| **18** | | | | | |
| **19** | | | | | |

3. Is this pattern familiar? Add 0's where you don't use those Tiles.
4. If you want to tile a line segment of length m, how many square Tiles will you need?

**2-Dimensional Tiling…**

When we tile something in 2-D using our special Tiles, we care about their height of $2^i$ and their width of $2^i$. This total square area must be used to tile a square part of the rectangle we are trying to tile over.

Consider a rectangle of dimensions 9 and 10.

a. What is the greatest number of Tiles of size $2^i$ x $2^i$ can you use to tile this rectangle?
b. What is the least number of Tiles you can use to tile this rectangle?

## RULE OF 1

Suppose you are only allowed to use exactly one Tile of each size. What sizes of rectangles can you tile with this new condition?

Fill in the chart on the following page with a **1** on the rectangles that Rule of 1 allow you to create. *For example if you have a 4x4 shaped rectangle you can use 1 of the* $2^2$x$2^2$ *Tiles to tile this rectangle shape*.

## RULE of 2

Now suppose you are only allowed to use up to two squares of each size. What types of rectangles can you tile with this condition? To keep track of your work (or to see some patterns), use the table provided.

*For example if you have a 4x6 shaped rectangle you can use 1 of the* $2^2$x$2^2$ *Tiles and 2 of the* $2^1$x$2^1$ *to tile this rectangle shape*.

### Questions about 2-D Tilings…

1. What patterns do you see in the table?
2. If you want to tile a rectangle of dimensions h x w what rectangles can you tile given Rules 1 and 2?

| | | WIDTH | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| HEIGHT | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | 1 | | | | | | | | | | | | |
| | 2 | | | | | | | | | | | | |
| | 3 | | | | | | | | | | | | |
| | 4 | | | | 1 | | 2 | | | | | | |
| | 5 | | | | | | | | | | | | |
| | 6 | | | | | | | | | | | | |
| | 7 | | | | | | | | | | | | |
| | 8 | | | | | | | | | | | | |
| | 9 | | | | | | | | | | | | |
| | 10 | | | | | | | | | | | | |
| | 11 | | | | | | | | | | | | |
| | 12 | | | | | | | | | | | | |
| | 13 | | | | | | | | | | | | |
| | 14 | | | | | | | | | | | | |
| | 15 | | | | | | | | | | | | |
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| | 21 | | | | | | | | | | | | |
| | 22 | | | | | | | | | | | | |
| | 23 | | | | | | | | | | | | |
| | 24 | | | | | | | | | | | | |

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Directed by Joshua Zucker
Castilleja School

# Building Blocks

You have an unlimited supply of cubes. Each cube's side length is a number of the form $2^i$, where $i$ is any nonnegative integer. That is, there are cubes with edges of length 1 unit, 2 units, 4 units, 8 units, 16 units, and so on.

You wish to build a tower of a given exact height by stacking cubes one on top of another.

**Part 1: Building Three Towers**
You need to build three towers, of heights 10, 15, and 32.

1. What is the biggest number of cubes you could use to build the three towers?
2. What is the smallest number of cubes you could use to build the three towers?
3. What is the biggest possible total volume of the three towers?
4. What is the smallest possible total volume?
5. What is the largest possible total surface area for the three towers?
6. What is the smallest possible total surface area?

**Part 2: Limited Supplies for One Tower**
Now you have only $m$ cubes, of edge length $N_1, N_2, N_3, \ldots, N_m$. For simplicity we'll suppose they go in increasing order, so $N_1$ is the smallest and $N_m$ is the biggest. There may be ties (or they may even all be the same size). You use them all to build a single tower.

7. What is the height of this tower?
8. What is the total volume of this tower?
9. What is the maximum possible total surface area for this tower?
10. What is the minimum possible total surface area?

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**Part 3: Friendly Fun**

Your friend builds a tower using at most *n* blocks of each side length. For instance, if *n* is 3 then your friend selected blocks from a pile with edges of length 1, 1, 1, 2, 2, 2, 4, 4, 4, 8, 8, 8, and so on. Also, the largest block in your friend's tower has length *c,* which equals $2^k$ for some value of *k*. (You might enjoy visiting the "To Twos" table if you like this part.)

11. What is the largest possible number of cubes that could be in your friend's tower?

12. What is the smallest possible number of cubes in your friend's tower?

13. What is the largest possible height for your friend's tower?

14. What is the smallest possible height?

**Part 4: Covering the Field**

Now you have a rectangular field, on the ground, that needs to be covered by cubes. No more towers, just every spot covered by a cube, and no cubes sticking out past the edges.

15. The field is 9 units long and 8 units wide. What is the largest number of cubes you can use to cover it?

16. What is the smallest number of cubes you can use to cover it?

17. Now you're allowed to use only one cube of each size. What rectangular fields can you cover?

18. New rules let you use up to two cubes of each size. What rectangular fields can you cover?

19. The field is *L* units long and *W* units wide. What is the smallest number of cubes needed to cover the field? Hint: start with small examples and work your way up in an organized table.

20. Write a formula expressing your solution method in the previous problem. You may find it useful to use the function $F(x, y)$ = the largest power of 2 that is less than or equal to both *x* and *y* in expressing your answer. Also, you may find it useful in writing your formula to assume that you have already figured out the fewest possible cubes for all smaller fields; feel free to use that function in your answer as well.