CHROMATIC NUMBERS AND OTHER GEOMETRIC COMBINATORICS DELIGHTS

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An intergalactic expedition discovers a uniquely dazzling world. Every point of its universe has its own color so it is stunningly beautiful. The magnanimous inhabitants of the world are more than willing to give their magic paints to the Earthlings to take back and paint our own space. There is only one little complication: if any two points of the same color are exactly one mile apart an explosive reaction is triggered. Still, people from Earth want to use the opportunity. Of course, due to extremely limited space on their starship, they have to take the minimal number of colors which would allow them to paint all our space without causing any explosions. What would that number be? We'll try to help them decide – and will discover some very interesting things along the way.

To begin with let's think of some intricate coloring schemes:

1. Is it possible to paint every point of a plane with one of three colors so that all three colors are used and every line of the plane consists of points of exactly two colors?

2. Is it possible to color each point of a plane with one of two colors in such a way that no two points exactly a unit distance apart are of the same color? What if we use three colors instead? Four colors?

The minimal number of colors which are needed in order to paint all points of the Euclidean *n*-space $\Re^n$ is called the *chromatic number* of the space; it is usually denoted by $\chi(\Re^n)$ or $\chi(n)$.

Well, perhaps dealing with the entire plane is too difficult. Let's look at something not as big, say, just one circle.

3. Is it possible to color each point on a circle either red or blue in such a way that no three points of the same color form an isosceles triangle? What if instead of just two colors you can use three different colors? Four colors? 1,000,000 colors?

Some (dis?)similar problems:

A. A.1 Is it possible to split the natural numbers into two sets *A* and *B* such that the sum of two distinct elements of *A* belongs to *B* and vice-versa?

A.2 Suppose that the set of all natural numbers is split into two sets *B* and *R*. We'll call the elements of *B* "blue", and the elements of *R* "red". Must there be integers *x*, *y* such that either all four numbers *x*, *y*, *x* + *y*, and *xy* are red, or all four of them are blue?

A.3 Suppose that natural numbers are partitioned into finitely many pieces:

$$\mathbb{N} = A_1 \cup A_2 \cup \ldots \cup A_n$$

(i.e., every integer is colored by one of n colors). Must there be integers *x*, *y* such that all four numbers *x*, *y*, *x* + *y*, and *xy* are of the same color?[1]

B. B.1 If 5 points lie in a plane so that no 3 points form a straight line, prove that four of the points will always form a convex quadrilateral.

B.2 If 9 points lie in a plane so that no 3 points are collinear, prove that 5 of the points form a convex pentagon.

B.3 If the number of points that lie in the plane is $1+2^{n-2}$ (where $n \geq 3$), and no 3 of them are collinear, can one always select *n* points so that they form a convex *n*-sided polygon?[2]

All the problems above belong to the part of mathematics called

Ramsey Theory.

Frank Ramsey, an English mathematician, economist and philosopher, proved his famous theorem in 1928. It says that if a number of objects in a set is sufficiently large and each pair of objects has one of a number of relations, then there is always a subset containing a certain number of objects where each pair has the same relation. Ramsey theory is concerned with finding just how large is sufficient.

To be a little more precise, we can look into a problem of finding *Ramsey Numbers.* A slightly different way to state Ramsey's theorem is to say that in any coloring of the edges of a sufficiently large complete graph, one will find monochromatic complete subgraphs. For two colors, Ramsey's theorem states that for any pair of positive integers (r,s), there exists a least positive integer R(r,s) such that for any complete graph with R(r,s) vertices, whose edges are colored *red* or *blue*, there exists either a complete subgraph with *r* vertices which is entirely blue, or a complete subgraph with *s* vertices which is entirely red.

Ramsey numbers are very hard to calculate. Only few Ramsey numbers are known so far:

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[1] This problem was posed by N. Hindman in 1979. It's still open.

[2] It is known that if there are sufficiently many points then it's possible to find n points forming a convex polygon. It is not known whether or not $1+2^{n-2}$ is a sufficiently large number. This number was conjectured by Erdös in 1934.

R(3,3) = 6;   R(3,4) = 9;   R(3,5) = 14;   R(3,6) = 18;   R(3,7) = 23;
R(3,8)=28;   R(3,9) = 36;   R(4,4) = 18;   R(4,5) = 25.

It is also known that $43 \le R(5,5) \le 49$, and $102 \le R(6,6) \le 165$, but nobody knows these two numbers exactly. In fact, Erdös used to say that if Aliens invade the Earth and threaten to obliterate it in a year's time unless human beings find R(5,5), w e could possibly avoid the obliteration by putting the world's best minds and fastest computers to the task. But if the aliens demanded that we find R(6,6) within a year, we would have no choice but to launch a pre-emptive attack.

Let's go back to problem 3. Let's cut the circle and straighten it up. If three points on the circle formed an isosceles triangle, what would these three points look like on this straight line?

Let's consider the following statement:

If all integers of a number line are colored, each with one of two colors, there must be three *monochromatic* (this means 'of the same color') numbers forming an arithmetic progression.

Can we prove it? Do we really need to color all integers? How many would suffice?   What if we use three different colors? Would we be able to find three monochromatic numbers forming an arithmetic progression in this case? What if we use 10 different colors and wish to find 23 monochromatic numbers forming an arithmetic progression?

The answer to these questions can be found in the following celebrated theorem.

Van der Waerden's Theorem: For any given positive integers *r* and *k*, there is some number *N* such that if the integers {1, 2, ... , *N*} are colored, each with one of *r* different colors, then there are at least *k* monochromatic integers forming an arithmetic progression.

The least such *N* is the *Van der Waerden Number W*(*r*, *k*). The current record for an upper bound belongs to Timothy Gowers (a Fields medallist); he proved that

$$W(r,k) \le 2^{2^{r^{2^{2^{k+9}}}}}$$

But it is an **open problem** to find the exact values of *W*(*r*, *k*) for most values of *r* and *k*, or even to reduce an upper bound (the Gower's bound is way bigger than actual value – for example, W(2, 3) = 9, and W(3, 3) – try to prove it).

Some closing remarks

The Chromatic Number problem belongs to *geometric combinatorics* (a.k.a. *combinatorial geometry*) which is a relatively new and rapidly growing branch of mathematics. It deals with geometric objects described by a finite set of building blocks, for example, bounded polyhedra and the convex hulls of finite sets of points. Other examples include arrangements and intersections of various geometric objects. Typically, problems in this area are concerned with finding bounds on a number of points or geometric figures that satisfy some conditions, or make a given configuration "optimal" in some sense.

Geometric combinatorics has many connections to linear algebra, discrete mathematics, mathematical analysis, and topology, and it has applications to economics, game theory, and biology, to name just a few.

Problems encountered within geometric combinatorics come in various forms; some are easy to state. Nevertheless, there are lots of problems that are extremely hard to solve, including a great many that remain open despite the efforts of some leading mathematicians. In particular, Borsuk's problem[3] has not yet been completely solved; to this day, the kissing numbers[4] are known only for dimensions 1, 2, 3, 4, 8, and 24 (these numbers are 2, 6, 12, 24, 240, and 196,560, respectively; surprisingly, the latest one found was the kissing number for $\Re^4$); as for the chromatic numbers, even $\chi(\Re^2)$ has not been found yet!

[3] To find the least number of subsets of smaller diameter in which a set of an *n*-dimensional space could be partitioned.

[4] Two spheres that have exactly one common boundary point are called *kissing* spheres. The largest number of equal spheres kissing a sphere of the same size is called the *kissing number*.