Graph theory III

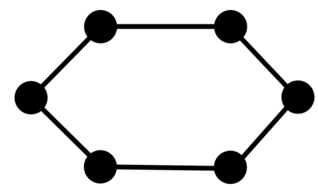
Berkeley Math Circle, Beginners

October 15, 2013

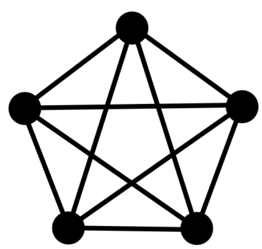
Reminders

A *graph* consists of *vertices* and *edges*. You can think of vertices as points, and edges are lines that connect some pairs of points. Here are some examples:

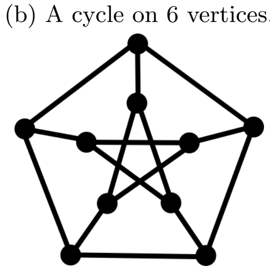
(a) A path on 4 vertices.



(b) A cycle on 6 vertices.



(c) The complete graph on 5 vertices.



(d) The Petersen graph.

Figure 1: Examples of graphs.

Although the figures above are all nice and symmetric, we do not care how a graph is drawn—the only thing that matters is whether there exists an edge between a pair of vertices or not. An edge can be a straight line, a curve, or anything you want, as long as it connects the two vertices. A pair of vertices which are connected by an edge are called *adjacent*.

Simple graphs are ones with no self-loops, and no multiple edges between two vertices. Also, we will only be considering *undirected* graphs: there is no distinction between the two vertices connected by an edge. (There are also *directed* graphs, where the edges point from one vertex to another.)

The *degree* of a vertex is the number of edges going out of it.

A *vertex coloring* of a graph is an assignment of colors to each vertex of the graph in such a way that no two adjacent vertices have the same color. An interesting question in general is to find the minimum number of colors needed to color a graph. This minimum number is called the *chromatic number* of the graph.

Problems

1. **(Petersen graph)** How many colors do you need to color the Petersen graph (see page 1)?

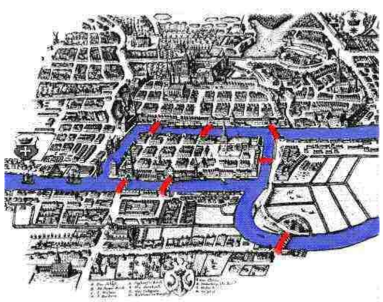
2. **(People at a party)** Consider a party of 5 people. Is it true that there is always a group of three who either all know each other or are all strangers to each other? What if the party consists of 6 people?

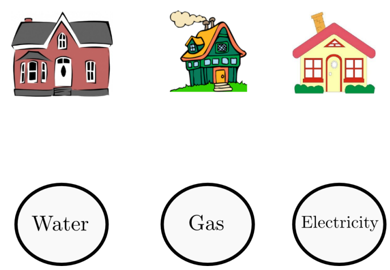
3. **(Seven bridges of Konigsberg)** A map of 18th century Konigsberg can be seen below; the city had seven bridges at the time. The citizens of Konigsberg used to spend Sunday afternoons walking around their city.

   (a) Is it possible to do so while crossing each bridge exactly once?

   (b) Suppose you are allowed to add an extra bridge. Where would you build it so that it is now possible to cross each bridge exactly once?

   (c) Suppose you live in the central island, and you are allowed to build two new bridges. Your goal is to build them so that it is now possible to cross each bridge exactly once, starting from the central island and ending at the central island. Can you do it?





4. **(Eulerian paths and cycles)** Now consider the graphs on page 1.

   (a) Is it possible to walk along the graph, so that each edge is crossed exactly once? (If this is possible, such a walk is called an *Eulerian path.*)

   (b) What if the starting and the ending vertices have to be the same? (If this is possible, such a walk is called an *Eulerian cycle.*)

5. **(Three houses, three utilities)** Above you see three houses, and the locations of three utilities: water, gas, and electricity. Each house needs to be connected to each utility by a road. However, the neighbors don't really like each other, so they want the roads to not cross each other. Is this possible?

6. **(Planar graphs)** A graph which can be drawn in the plane without crossing edges is called a *planar graph*. Is the complete graph on 4 vertices planar? How about the complete graph on 5 vertices? And the other graphs on page 1?

7. **(Perfect matchings)** Consider a set of 5 girls (Angela, Bella, Christina, Daniella, and Eva) and 5 boys (Abe, Bo, Cam, Dan, and Ed). Some of them like each other, others don't. In the two scenarios depicted below, an edge between a boy and a girl means that they like each other. Is it possible to match everyone up to form 5 pairs, in each of which the boy and the girl like each other?