Probability and Random Walks 2
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Recall: A **probability space** is a set of events, each of which have a certain probability. A **random variable** X is a variable that assigns values to each event in the probability space: For example, the sum of three coin flips or two rolls of a die.

This Time: Two useful tools for analyzing random variables are the **expected value** and the **variance**.

EXPECTED VALUE:

The **expected value** of a random variable is defined as a sum:

$$\boxed{E(X) = P(\text{Event } 1)X(\text{Event } 1) + P(\text{Event } 2)X(\text{Event } 2) + \cdots}$$

Problem 1: Find the expected value of the following random variables:

(a) The result of a single coin flip.

(b) The roll of a single $6-$sided die.

(c) For a roll of a six-sided die, let X be the random variable that takes the value 1 if you roll a 6 and 0 otherwise.

(d) For a roll of two six-sided dice, let X be the random variable which equals the total number of sixes rolled. Hint: You can draw the entire event space.

(e) In a game, you have $1/2$ chance of winning ten dollars, $1/4$ chance of losing two dollars, and $1/4$ chance of winning six dollars. Find the expected value of your winnings.

(f) In another game, you flip three coins. You win ten dollars if all three are heads, and otherwise you win nothing. Find the expected value of your winnings.

The expected value satisfies the following **rule:**

$$\boxed{E(X+Y) = E(X) + E(Y)}$$

Problem 2: Find the expected value of the following random variables:

(a) The sum of two coin flips. The sum of three coin flips. Can you find a pattern?

(b) The sum of two rolls of a six-sided die. Three rolls? More?

(c) For a roll of three six-sided dice, let X be the random variable which equals the total number of sixes rolled.

Problem 3: Find the expected position of a random walk in one dimension after n moves.

VARIANCE:
The **variance** of a random variable X is defined using the expected value:

STEP 1: Write $\mu = E(X)$.

STEP 2: The variance is defined: $\boxed{V(X) = E((X-\mu)^2)}$.

Problem 4: Find the variance of all of the random variables in problem 1.

Random variables X and Y are called **independent** if knowing the value of one doesn't affect what you know about the other: They have no relationship to each other. Examples are: The results of two separate coin flips, the results of two separate rolls of a die, or the steps in a random walk.

Properties of independent random variables: If X and Y are independent:

$$\boxed{E(XY) = E(X)E(Y)}$$

$$\boxed{V(X+Y) = V(X) + V(Y)}$$

Problem 5: Prove the two properties above.

Problem 6: Find the variance in flipping n coins.

Problem 7: Find the variance in the position of a random walk after n steps.

Problem 8: Find the variance and expected value of a two-dimensional random walk after n steps. What about a $d-$dimensional walk?

Below: A self-avoiding random walk