Patterns,
Geometry,
and Induction

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■ Patterns and Expressions

1. What is the formula for the n^{th} odd number? The $(n + 1)^{\text{st}}$ odd number?
2. Find and prove a formula for the sum $1 + 2 + 3 + \cdots + n$.
3. Find and prove a formula for the sum of the first n odd numbers.
3. Use the image on the back to establish a formula for the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$.
4. Use the results of problem 3 to find the volume of a cone or pyramid.
5. Use the image on the back to establish a formula for the sum $1^3 + 2^3 + 3^3 + \cdots + n^3$.
6. Prove the Hockey Stick theorem (for Pascal's Triangle).
7. Use the Hockey Stick Theorem to prove…
   a) … that $1\cdot 2 + 2\cdot 3 + \cdots + (n-1)\cdot n = \frac{n\cdot(n+1)\cdot(n+2)}{3}$
   b) … that $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + (n-1)\cdot n\cdot (n+1) = \frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}{4}$
8. What is the greatest number of pieces you can get by making n straight cuts through a circular pizza?
9. What is the greatest number of pieces you can get by joining n points on a circle in every possible way?
10. What is the greatest number of space-regions you can get by placing n planes in space?
11. Can every odd number greater than 3 be written as the sum of a prime number and a power of 2?
12. Can every odd number greater than 2 be written as a prime plus twice a square?

■ Mathematical Induction

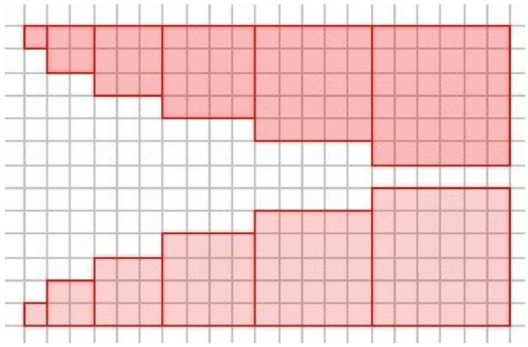
Mathematical induction is used to prove that a fact is true for all (natural number) values of n.

•Proving the fact works for the initial term (or terms) of the sequence is the *anchor*.

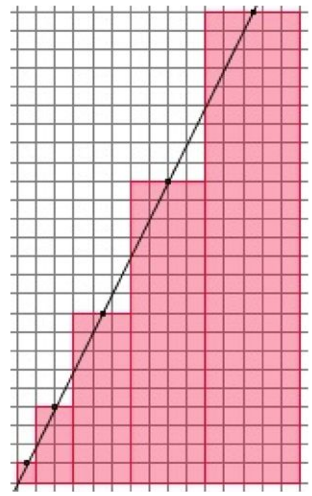
•Proving that if it works for $n - 1$, then it works for n is the inductive (or recursive) step.

•If you prove both, you are done.

13. Prove that $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n! = (n+1)! - 1$
14. Use induction to prove that $n^2 + (n+1)^2 + (n+2)^2$ is always divisible by 9.
15. Use induction to prove that $6^n - 1$ is always divisible by 5.
16. Use Induction to prove…
   a) … that $1\cdot 2 + 2\cdot 3 + \cdots + (n-1)\cdot n = \frac{n\cdot(n+1)\cdot(n+2)}{3}$
   b) … that $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + (n-1)\cdot n\cdot (n+1) = \frac{n\cdot(n+1)\cdot(n+2)\cdot(n+3)}{4}$
17. Find and prove a formula for $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n}$



Use this image to understand the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$



Use this image to understand the sum $1^3 + 2^3 + 3^3 + \cdots + n^3$

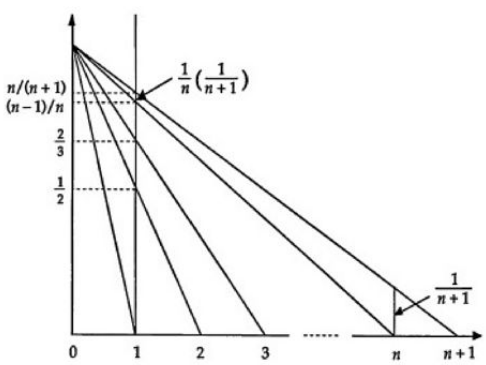
Let F_n be the n^{th} Fibonacci number, defined by $F_0 = 1,\ F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$.

18. Prove that $F_{k-1} \cdot F_{k+1} = {F_k}^2 + (-1)^k$.

19. Prove that $\sum_{k=0}^{n} {F_k}^2 = F_n \cdot F_{n+1}$

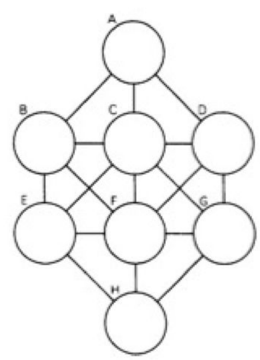
20. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$.

21. Prove that $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.



What is the least possible value of the smallest of 99 consecutive positive integers whose sum is a perfect cube?

(2014 California Math League, 14 January)



Place the digits 1–8 into the circles so that no neighbors are connected by the given line segments.