**Geometry with Complex Numbers**
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I. **Review of complex numbers.** A complex number $z$ can be written uniquely in the form $z = a + bi$ where $a, b$ are real numbers, called the *real* and *imaginary part* of $z$. We identify such a $z$ with the vector $(a, b)$ in the plane and we identify a real number $a$ with the complex number $z = a + 0i$. We add complex numbers as we do with vectors, i.e. $(a + bi) + (c + di) = (a + c) + (b + d)i$. To multiply two complex numbers we treat both factors as polynomials of a variable $i$ and use the formula $i^2 = -1$:

$$(a + bi) \cdot (c + di) = ac + (bc + ad)i + bdi^2 = (ac - bd) + (bc + ad)i.$$

All the standard algebraic identities work for complex numbers (the complex numbers form a *field*).

The number $|z| := r = \sqrt{a^2 + b^2}$ (the length of the corresponding vector) is called the *absolute value* of $z$. We call the number $\overline{z} := a - bi$ (reflection of the corresponding vector w.r.t. the real axis) the *conjugate* of $z$; we then have $\overline{zw} = \overline{z} \cdot \overline{w}$, $\overline{z + w} = \overline{z} + \overline{w}$ and $z\overline{z} = |z|^2$, so $1/z = \overline{z}/|z|^2$ (the last formula allows us to divide complex numbers). Note that the real part $a = (z + \overline{z})/2$ and the imaginary part $b = (z - \overline{z})/2i$. The real numbers are characterized by $z = \overline{z}$ and the numbers of absolute value 1 (the set of such numbers is called the *unit circle*) are characterized by $z^{-1} = \overline{z}$.

II. **Polar form and geometric interpretation of multiplication.** Let $\varphi$ be the angle between the real axis and the vector corresponding to $z = a + bi$ (counterclockwise) and let $r = |z|$. Then $a = r\cos\varphi$ and $b = r\sin\varphi$, hence

$$z = r(\cos\varphi + i\sin\varphi).$$

This is called the *polar form* of $z$. It is useful for multiplying complex numbers: by standard trig identities we have

$$r(\cos\varphi + i\sin\varphi) \cdot R(\cos\psi + i\sin\psi) = rR(\cos(\varphi + \psi) + i\sin(\varphi + \psi)),$$

i.e. the absolute values multiply and the angles add. One can interpret this by saying that given $z$, the transformation $w \mapsto zw$ on the plane is the composition of a homothety (rescaling) by a factor of $r = |z|$ and a counterclockwise rotation around the origin by the angle $\varphi$.

*Problem* 1. Given are three disjoint squares $ABCD$, $BEFC$ and $EGHF$. Find the sum of the three angles $CAB$, $FAB$ and $HAB$.

*Problem* 2. Describe how to construct a regular pentagon using ruler and compass. *Hint: solve the algebraic equation* $z^5 = 1$.

*Problem* 3. Interpret the transformation $z \mapsto 1/z$ using inversion.

III. **Triangles.** Given a triangle $ABC$, we treat the vertices $A$, $B$, $C$ as complex numbers. From the vector interpretation of complex numbers we have that the centroid of $ABC$ is $(A + B + C)/3$.

*Problem* 4. We build equilateral triangles on the sides of a given triangle (on the outside, so that the three new triangles share only edges with the given triangle). Prove that the centroids of the three new triangles form an equilateral triangle. (This is often called Napoleon's Theorem).

What about the *circumcenter* and *orthocenter*? First, we note that whenever in a given problem there is a circle, we should consider making that circle the unit circle and then exploiting the relation $z\overline{z} = 1$. So, if a problem about a triangle $ABC$ involves the circumscribed circle and the circumcenter $O$, we like to assume that $A\overline{A} = B\overline{B} = C\overline{C} = 1$, so $O = 0$.

*Problem* 5. In the above setup, find a formula for the orthocenter $H$ of $ABC$. *Hint: use the Euler line.*

IV. **Lines.** To find the intersection point of two lines, it is useful to write the equations for the lines in terms of $z$ and $\overline{z}$.

*Problem* 6. Find the equation in $z$ and $\overline{z}$ saying that a complex number $z$ lies on a line $ab$. Find a condition saying that $zw \perp ab$.

*Problem* 7. Verify that in Problem 5 we have $AH \perp BC$ using the previous problem.

V. **My favorite trick.** Whenever we have a circle *inscribed* in a polygon, it is fruitful to treat as input not the vertices, but the points where the sides touch the circle. The following problem allows us then to express the vertices in terms of these tangency points:

*Problem* 8. Let $a$ and $b$ lie on the unit circle and let $p$ be the point of intersection of the tangent lines to the unit circle at $a$ and $b$. Prove that

$$p = \frac{2ab}{a+b}.$$

*Hint: use Problem 3.*

*Problem* 9. A quadrilateral $ABCD$ is circumscribed on a circle and the sides $AB$ and $CD$ are parallel. The side $DA$ is tangent to the inscribed circle at $E$. Point $F$ is the reflection of $A$ with respect to $B$. The line tangent to the inscribed circle, passing through $F$ and distinct from $AB$ touches the circle at $G$. Prove that $C$, $E$ and $G$ are colinear.

**More problems.** In the following problems, you are advised to reduce the given problem to a simple algebraic manipulation using complex numbers. You don't have to finish this manipulation as long as you're convinced you can do it.

*Problem* 10. On the sides of a parallelogram we build four squares. Prove that the centers of these squares form a square.

*Problem* 11. Given a triangle $ABC$; let $O$ be the circumcenter, $H$ the orthocenter, $R$ the radius of the circumscribed circle. Let $A'$, $B'$, $C'$ be the reflections of $A$, $B$, $C$ with respect to the opposite sides. Prove that $A'$, $B'$ and $C'$ are colinear if and only if $|OH| = 2R$.

*Problem* 12. Prove that if a quadrilateral is circumscribed on a circle, then the diagonals and the segments connecting opposite tangency points all pass through one point.

*Problem* 13. A point $A$ lies inside a circle. Given a chord $ZW$ of the circle passing through $A$, let $P$ be the intersection point of the tangent lines to the circle at $Z$ and $W$ (if it exists). Prove that the locus of the points $P$ with fixed $A$ and a moving chord is a line.

*Problem* 14. Points $P$, $A$, $B$, $C$ are distinct and lie on a circle. Prove that the projections of $P$ on the lines $AB$, $BC$, $CA$ are colinear (this line is called the Simson line).

Finally, a problem without geometry:

*Problem* 15. Calculate the sum

$$\sum_{k\geq 0} \binom{n}{3k}.$$

*Hint: use the number* $\xi = (1 + i\sqrt{3})/2$.