Solving the cubic via circulant matrix theory

Exercise 1. Let $p(x) = x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_0$. Show that the change of variables

$$x = y - \frac{\alpha_{n-1}}{n}$$

eliminates the term of degree $n-1$.

The general cubic polynomial is

$$p(x) = x^3 + \alpha x^2 + \beta x + \gamma,$$

and the general 3×3 circulant matrix is

$$C = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}.$$

Thanks to the above exercise, we may assume $\alpha = 0$. The objective is to obtain expressions for the roots of $p(x) = x^3 + \alpha x^2 + \beta x + \gamma$. We want to find a, b, and c so that p is the characteristic polynomial of C.

Exercise 2. Verify that $\det(tId - C) = (t-a)^3 - 3bc(t-a) - (b^3 + c^3)$.

Thus, to equate p with $\det(tId - C)$, we choose $a = 0$ and

$$\begin{aligned} 3bc &= -\beta, \\ b^3 + c^3 &= -\gamma \end{aligned} \qquad .$$

Exercise 3. Solve the above system for b and c.

Exercise 4. Use circulant matrix theory to write the roots of p (i.e. the eigenvalues of C) and thereby finish the cubic solution.

The quartic case may be obtained by following the same strategy as above. Note that by employing this strategy, one may assume from the start that the diagonal entries of the initial 4×4 circulant are zero.