11th Bay Area Mathematical Olympiad

BAMO-12 Exam

February 24, 2009

The time limit for this exam is 4 hours. Your solutions should be clearly written arguments. Merely stating an answer without any justification will receive little credit. Conversely, a good argument which has a few minor errors may receive substantial credit.

Please label all pages that you submit for grading with your identification number in the upper-right hand corner, and the problem number in the upper-left hand corner. Write neatly. If your paper cannot be read, it cannot be graded! Please write only on one side of each sheet of paper. If your solution to a problem is more than one page long, please staple the pages together.

The five problems below are arranged in roughly increasing order of difficulty. Few, if any, students will solve all the problems; indeed, solving one problem completely is a fine achievement. We hope that you enjoy the experience of thinking deeply about mathematics for a few hours, that you find the exam problems interesting, and that you continue to think about them after the exam is over. Good luck!

Note that the five problems are numbered 3–7. This is because BAMO-8, the middle-school version, has four problems, numbered from 1 to 4. The two hardest problems of BAMO-8 are the first two problems of BAMO-12. So collectively, the problems of the two BAMO exams are numbered 1–7.

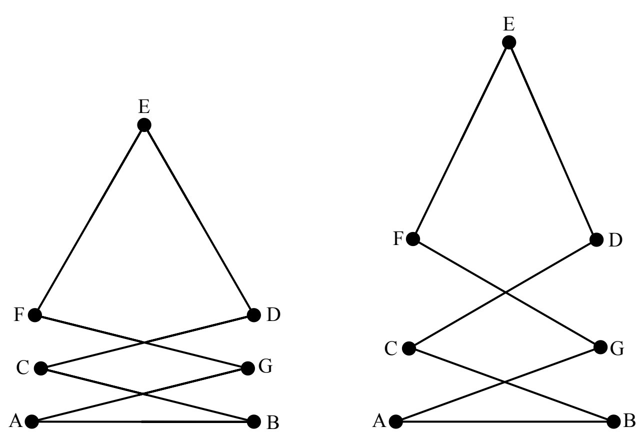
Problems

3 There are many sets of two different positive integers a and b, both less than 50, such that a^2 and b^2 end in the same last two digits. For example, $35^2 = 1225$ and $45^2 = 2025$ both end in 25. What are all possible values for the average of a and b?

For the purposes of this problem, single-digit squares are considered to have a leading zero, so for example we consider 2^2 to end with the digits 04, not 4.

4 Seven congruent line segments are connected together at their endpoints as shown in the figure below at the left. By raising point E the linkage can be made taller, as shown in the figure below and to the right. Continuing to raise E in this manner, it is possible to use the linkage to make A, C, F, and E collinear, while simultaneously making B, G, D, and E collinear, thereby constructing a new triangle ABE.

Prove that a regular polygon with center E can be formed from a number of copies of this new triangle ABE, joined together at point E, and without overlapping interiors. Also find the number of sides of this polygon and justify your answer.



5 A set S of positive integers is called *magic* if for any two distinct members of S, i and j,

$$\frac{i+j}{GCD(i,j)}$$

is also a member of S. The GCD, or greatest common divisor, of two positive integers is the largest integer that divides evenly into both of them; for example, $GCD(36,80) = 4$.

Find and describe all finite magic sets.

6 At the start of this problem, six frogs are sitting at each of the six vertices of a regular hexagon, one frog per vertex. Every minute, we choose a frog to jump over another frog using one of the two rules illustrated below. If a frog at point F jumps over a frog at point P, the frog will land at point F' such that F, P, and F' are collinear and:

- using Rule 1, $F'P = 2FP$.
- using Rule 2, $F'P = FP/2$.

Rule 1 *Rule 2*

It is up to us to choose which frog to take the leap and which frog to jump over.

(a) If we only use Rule 1, is it possible for some frog to land at the center of the original hexagon after a finite amount of time?

(b) If both Rule 1 and Rule 2 are allowed (freely choosing which rule to use, which frog to jump, and which frog it jumps over), is it possible for some frog to land at the center of the original hexagon after a finite amount of time?

7 Let $\triangle ABC$ be an acute triangle with angles α, β, and γ. Prove that

$$\frac{\cos\alpha}{\cos(\beta-\gamma)} + \frac{\cos\beta}{\cos(\gamma-\alpha)} + \frac{\cos\gamma}{\cos(\alpha-\beta)} \geq \frac{3}{2}.$$

You may keep this exam. **Please remember your ID number!** Our grading records will use it instead of your name.

You are cordially invited to attend the **BAMO 2009 Awards Ceremony**, which will be held at the Mathematical Sciences Research Institute, from 11–2 on Sunday, March 8. This event will include lunch, a mathematical talk by Francis Su of Harvey Mudd, and the awarding of dozens of prizes. Solutions to the problems above will also be available at this event. Please check with your proctor for a more detailed schedule, plus directions.