Berkeley Math Circle
Monthly Contest 8
Due May 6, 2014

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 8, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Define a "word" to be a string of at most ten letters taken from the English alphabet. (The letters do not have to be distinct.) Prove that the number of "words" is divisible by 27.

2. A pool can be filled through four different pipes. If the first and second pipes are operating, it takes 2 hours to fill the pool. With the second and third pipes it takes 3 hours, and with the third and fourth it takes 4 hours. How long does it take to fill the pool if the first and fourth pipes are operating?

3. Some soldiers are standing in a line in the east-west direction, each of them facing north. Their officer commands, "Right face!" They should now all be facing east, but, as they are at the very beginning of their military career, some of them get the order wrong and turn to the west. Every soldier who is then facing his neighbor immediately concludes that he has made a mistake and performs a 180° turn within 1 second. The process continues, so that any two soldiers who are now facing each other perform 180° turns within 1 second. Prove that after some time, the soldiers will stop moving.

4. A 23×23 square is divided into smaller squares of dimensions 1×1, 2×2, and 3×3. What is the minimum possible number of 1×1 squares?

5. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Let M and N be the respective midpoints of AB and CD. Diagonals AC and BD meet at P, and lines AD and BC meet at Q. Prove that M, N, P, and Q are collinear.

6. Let $a_1 = 5$ and $a_{n+1} = a_n^3 - 2a_n^2 + 2$ for all $n \geq 1$. Prove that if p is a prime divisor of $a_{2014} + 1$ and $p \equiv 3 \bmod 4$, then $p = 3$.

7. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) - f(x) = (x + f(y))^4 - x^4$$

for all $x, y \in \mathbb{R}$.