Berkeley Math Circle
Monthly Contest 4 – Solutions

1. Is there a positive integer k such that
$$(\cdots((4\underbrace{!)!)!\cdots)!}_{k} > (\cdots((3\underbrace{!)!)!\cdots)!}_{k+1}?$$

   Solution. The answer is no. Since $3! = 6$, we have
$$(\cdots((3\underbrace{!)!)!\cdots)!}_{k+1} = (\cdots((6\underbrace{!)!)!\cdots)!}_{k} > (\cdots((4\underbrace{!)!)!\cdots)!}_{k}$$
   where the last step follows by using the obvious lemma that if $x > y$ then $x! > y!$ (for positive integers x and y) k times.

2. Two regular polygons are said to be *matching* if the double of the interior angle of one of them equals the triple of the exterior angle of the other. Find all pairs of matching polygons.

   Solution. The answers, expressed in terms of numbers of sides, are $(3, 9)$, $(4, 6)$, $(5, 5)$, and $(8, 4)$. They are easily verified to be matching.

   To prove that there are no others, first note that if the first polygon has at most five sides, the second one necessarily has one of the numbers of sides necessary to make one of the first three matching pairs. The second polygon cannot be a triangle (since then the first polygon would have an interior angle of $180°$), and if it is a square or pentagon, we recover the last two matching pairs. So if there are any matching pairs other than the four listed, both polygons must have at least six sides. But then the interior angle of the first is at least $120°$ and the exterior of the second is at most $60°$, which is impossible.

3. A natural number n is chosen between two consecutive square numbers. The smaller square is obtained by subtracting k from n, and the larger one is obtained by adding ℓ to n. Prove that the number $n - k\ell$ is the square of an integer.

   Solution. Let $n - k = x^2$ and $n + \ell = (x+1)^2$. Then $k = n - x^2$ and $\ell = (x+1)^2 - n$. We express everything in terms of n and x:
$$\begin{aligned} n - k\ell &= n - (n - x^2)((x+1)^2 - n) \\ &= n + (x^2 - n)((x^2 + 2x + 1 - n) \\ &= n + (x^2 - n)((x^2 - n) + 2x + 1) \\ &= n + (x^2 - n)^2 + 2x(x^2 - n) + x^2 - n \\ &= (x^2 - n)^2 + 2x(x^2 - n) + x^2 \\ &= (x^2 - n + x)^2. \end{aligned}$$
   This is the square of an integer.

4. A positive integer is written in each cell of an 8×8 table so that each entry is the arithmetic mean of some two of its neighbors. Find the maximum number of distinct integers that may appear in the table.

   Solution. First consider the minimum number m in the table. If it appears in some cell A, two neighboring cells B, C must also contain m because there is no other way for m to be the arithmetic mean of two numbers in the table. Since B cannot neighbor C, it must have another neighbor D in addition to A that contains m.

   So the minimum number in the table appears at least four times. Likewise, the maximum number appears at least four times. We can now find 8 cells that contain at most 2 distinct numbers. The remaining 56 cells of course contain at most 56 distinct numbers. So there are at most 58 distinct numbers in the table. The diagram below shows that this bound can be achieved.

1	1	21	22	37	38	58	58
1	1	20	23	36	39	58	58
3	2	19	24	35	40	57	56
4	5	18	25	34	41	54	55
7	6	17	26	33	42	53	52
8	9	16	27	32	43	50	51
11	10	15	28	31	44	49	48
12	13	14	29	30	45	46	47

5. Prove that

$$-\frac{1}{2} \le \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} \le \frac{1}{2}$$

for all real numbers x and y.

Solution 1. Since x and y can be replaced by their negatives, it suffices to prove the right-hand inequality. On cross-multiplication, this is

$$\begin{aligned} 0 &\overset{?}{\le} (x^2+1)(y^2+1) - 2(x+y)(1-xy) \\ &= x^2y^2 - 2x^2y + x^2 - 2xy^2 + 2x + y^2 + 2y + 1 \\ &= (xy - x - y - 1)^2. \end{aligned}$$

Solution 2. Let $x = \tan\alpha$ and $y = \tan\beta$. Then

$$\begin{aligned} \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)} &= \frac{(\tan\alpha + \tan\beta)(1 - \tan\alpha\tan\beta)}{\sec^2\alpha\sec^2\beta} \\ &= \cos\alpha\cos\beta(\tan\alpha+\tan\beta)\cdot\cos\alpha\cos\beta(1-\tan\alpha\tan\beta) \\ &= (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= \sin(\alpha+\beta)\cos(\alpha+\beta) \\ &= \frac{1}{2}\sin(2\alpha + 2\beta), \end{aligned}$$

which lies between $-1/2$ and $1/2$ inclusive.

6. Let $ABCDEF$ be a hexagon circumscribing a circle ω. The sides AB, BC, CD, DE, EF, FA touch ω at U, V, W, X, Y, and Z respectively; moreover, U, W, and Y are the midpoints of sides AB, CD, and EF, respectively. Prove that UX, VY, and WZ are concurrent.

Solution. Since U is the midpoint of AB, we have $ZA = AU = UB = BV$ and so (letting O be the center of ω) $\triangle OZA \cong \triangle OUA \cong \triangle OUB \cong \triangle OVB$. Thus arcs ZU and UV are equal, and so UX is the bisector of VXZ. Similarly, VY and WZ are the bisectors of the other two angles of $\triangle VXZ$, so these three lines are concurrent.

7. Let a and b be positive integers. Define a sequence $x_0, x_1, x_2, \ldots$ by $x_0 = 0$, $x_1 = 1$, and $x_{n+2} = ax_{n+1} + bx_n$ for $n \ge 0$. Prove that

$$\frac{x_{m+1}x_{m+2}\cdots x_{m+n}}{x_1x_2\cdots x_n}$$

is an integer for all positive integers m and n.

Solution. Let $c_{m,n}$ be the quotient in the problem. Inspired by the similarity to $\binom{m+n}{m}$, we seek a relation between $c_{m,n}$, $c_{m-1,n}$, and $c_{m,n-1}$. First we prove the addition identity:

Lemma 1. *For $m \ge 1$ and $n \ge 0$,*

$$x_{m+n} = x_m x_{n+1} + b x_{m-1} x_n.$$

Proof. Induction on m. The cases $m = 1$ and $m = 2$ are trivial. Adding b times the equation for $m = m_0$ to a times the equation for $m = m_0 + 1$ yields the equation for $m = m_0 + 2$. □

We can now expand

$$\begin{aligned} c_{m,n} &= \frac{x_{m+1}\cdots x_{m+n-1}x_{m+n}}{x_1x_2\cdots x_n} \\ &= \frac{x_{m+1}\cdots x_{m+n-1}(x_m x_{n+1} + b x_{m-1} x_n)}{x_1x_2\cdots x_n} \\ &= x_{n+1}c_{m-1,n} + b x_{m-1} c_{m,n-1}. \end{aligned}$$

We can now solve the problem by induction on $m + n$. The base cases $c_{m,0} = c_{0,n} = 1$ are integers, and the foregoing shows that if $c_{m-1,n}$ and $c_{m,n-1}$ are integers, so is $c_{m,n}$.

Remark. Solutions involving counting the prime factors of the numerator and denominator of $c_{m,n}$ can be found, but they are considerably messier than the foregoing, especially when a and b share prime factors.