Berkeley Math Circle
Monthly Contest 4
Due January 7, 2014

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 4, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A triangle, two of whose sides are 3 and 4, is inscribed in a circle. Find the minimal possible radius of the circle.

2. Determine, with proof, whether or not there exist positive integers a, b, and c such that
$$ab + bc = ac \quad \text{and} \quad abc = 10!.$$
Remark. 10! denotes the factorial $1 \cdot 2 \cdot 3 \cdots 10$.

3. A building has the plan of a 5×5 grid of rooms, each of which has a door in each of its four walls: thus there are 20 doors leading to the outside. The doors are to be opened and closed so that every room has exactly 3 open doors leading from it. Determine the minimum and maximum number of doors to the outside that may be left open.

4. Prove that for each $n \geq 1$, there is a number N having n digits, each of which is either 1 or 2, such that N is divisible by 2^n.

5. Let $n \geq 1$ be an integer. How many ways can the rectangle having vertices $(0, 0)$, $(n, 0)$, $(n, 1)$, $(0, 1)$ be dissected into $2n$ triangles, all vertices of which have integer coordinates?
Remark. The triangles are considered as positioned on the coordinate plane; in particular, tilings related by rotation and reflection are considered distinct.

6. Show that
$$a + b + c + \sqrt{3} \geq 8abc \left(\frac{1}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \right)$$
for all positive real numbers a, b, c satisfying $ab + bc + ca \leq 1$.

7. The incircle of a triangle ABC touches the sides BC, CA, and AB at points D, E, and F, respectively. The circle passing through point A and tangent to BC at D intersects the line segments BF and CE at points K and L, respectively. The line through E parallel to DL and the line through F parallel to DK intersect at P. Let R_1, R_2, R_3, R_4 denote the respective circumradii of triangles AFD, AED, FPD, and EPD. Prove that $R_1 R_4 = R_2 R_3$.