Berkeley Math Circle
Monthly Contest 3 – Solutions

1. Fifty counters are on a table. Two players alternate taking away 1, 2, 3, 4, or 5 of them. Whoever picks up the last counter is the loser. Who has a winning strategy, the first player or the second?

   Solution. Note that if you make a turn and there is 1 counter left, you have won since the other player must pick up that counter.

   If you make a turn and there are 7 counters left, you can win: if your opponent picks up 1, 2, 3, 4, or 5 of them, you can respectively take 5, 4, 3, 2, or 1 of them to leave 1.

   Likewise, if you play and there are 13 counters left, you can in the same way play to leave 7 on your next turn.

   Continuing in this way, we see that the first player can win by removing 1 counter, leaving 49, and then playing on the succeeding turns to leave 43, 37, 31, 25, 19, 13, 7, and 1.

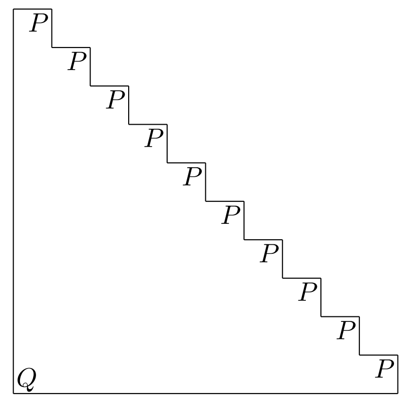
2. How many divisors does 2013^{13} have? (As usual, we count 1 and 2013^{13} itself as divisors, but not negative integers.)

   Solution. The prime factorization of 2013 is $3 \cdot 11 \cdot 61$, so

   $$2013^{13} = 3^{13} \cdot 11^{13} \cdot 61^{13}.$$

   A divisor of this number is found by choosing 0 to 13 of the factors 3 (there are 14 possible choices), 0 to 13 of the factors 11 (14 choices), and 0 to 13 of the factors 61 (14 choices). So the total number of divisors is $14 \cdot 14 \cdot 14 = 2744$.

3. Define an *n-staircase* to be the union of all squares of an $n \times n$ grid lying on or below its main diagonal. How many ways are there to divide a 10-staircase into 10 rectangles, each having a side of length 1? (Reflections are not included.)

   Solution. A 10-staircase has 10 "upper right corners" P, each of which must be the upper right corner of some rectangle, and no two of which can belong to the same rectangle. It also has a single lower left corner Q which must belong to the same rectangle as one of the ten points P. Since this rectangle has one side of length 1, it must be a 10×1 rectangle placed either vertically or horizontally along the long side of the staircase. The remainder of the figure is then a 9-staircase to be filled with 9 rectangles.

   We can then repeat the argument to find that one of the long sides of the 9-staircase must be filled by a 9×1 rectangle, leaving an 8-staircase. This continues until we reach the 1-staircase, a single square, which can be filled in only one way.

   The placement of the 10×1 rectangle is irrelevant because of the symmetry of the shape. But the 9×1 through 2×1 rectangles each involve a choice between two alternatives, so the number of tilings is $2^8 = 256$.

4. Let x, y, and z be real numbers such that $xyz = 1$. Prove that

   $$x^2 + y^2 + z^2 \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

   Solution. Replacing the 1 in the numerators of the fractions on the right by xyz, it suffices to prove that

   $$x^2 + y^2 + z^2 \geq yz + zx + xy \tag{1}$$

   which is true because

   $$x^2 + y^2 + z^2 - yz - zx - xy = \frac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2} \geq 0.$$

   (Alternatively, (1) is a consequence of Cauchy's Inequality or of the Rearrangement Inequality.)

5. Let $BCED$ be a cyclic quadrilateral. Rays CB and ED meet at A. The line through D parallel to BC meets ω at $F \neq D$, and segment AF meets ω at $T \neq F$. Lines ET and BC meet at M. Let K be the midpoint of BC, and let N be the reflection of A about M. Prove that points D, N, K, E lie on a circle.

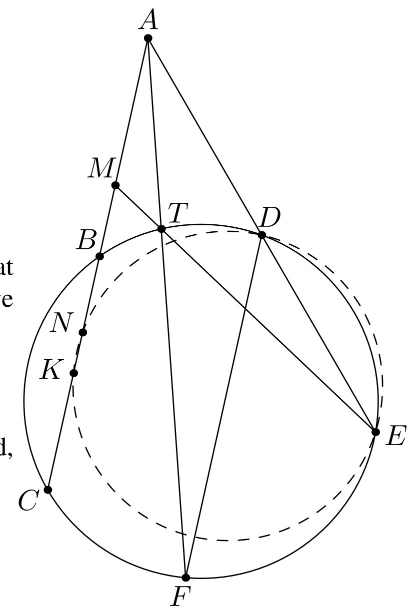
Solution. We have $\angle MAT = \angle DFT = \angle DET$, so $\triangle AMT \sim \triangle EMA$, giving $AM/MT = EM/AM$, so $AM^2 = ME \cdot MT$. But by Power of a Point, $ME \cdot MT = MB \cdot MC$. So

$$AM^2 = MB \cdot MC = (AB - AM)(AC - AM) = AB \cdot AC - AM(AB + AC) + AM^2,$$

that is,

$$AB \cdot AC = AM(AB + AC) = AM \cdot 2 \cdot AK = AN \cdot AK.$$

We derive that $AD \cdot AE = AN \cdot AK$, so D, E, K, and N are concyclic.



6. For each $n \geq 1$, determine (in closed form) the number of integers k such that

- $0 \leq k < 4^n$;
- k is a multiple of 3;
- The sum of the binary digits of k is even.

Solution 1. Let $a_{ij} = a_{ij}(n)$ denote the number of integers k, $0 \leq k < 4^n$, such that $k \equiv i \mod 3$ and the sum of the binary digits of k is congruent to j mod 2. Thus we have a decomposition of all 4^n of these numbers into six categories:

$$4^n = a_{00} + a_{01} + a_{10} + a_{11} + a_{20} + a_{21}.$$

Since equally many binary numbers in the range $0 \leq k < 4^n$ have even digit sum as odd, we have the relation

$$a_{00} + a_{10} + a_{20} = a_{01} + a_{11} + a_{21} = 2^{2n-1}.$$

Also, we know the number of integers in each of the congruence classes mod 3:

$$a_{00} + a_{01} = \frac{4^n + 2}{3}, \quad a_{10} + a_{11} = a_{20} + a_{21} = \frac{4^n - 1}{3}.$$

Lastly, we have the symmetries $a_{10} = a_{20}$, $a_{11} = a_{21}$ coming from the fact that moving the first binary digit of k to the end (assuming the representation to be zero-padded so that there are $2n$ digits) doubles the number, possibly subtracting $4^n - 1$ which is divisible by 3, while keeping the digit sum fixed. This is not enough information to compute a_{00}, but it does let us express all the other a_{ij}'s in terms of a_{00}:

$$\begin{aligned}
a_{01} &= \frac{4^n + 2}{3} - a_{00} \\
a_{10} = a_{20} &= \frac{1}{2}(2^{2n-1} - a_{00}) \\
a_{11} = a_{21} &= \frac{1}{2}(2^{2n-1} - a_{01}) = \frac{1}{2}\left(2^{2n-1} - \frac{4^n + 2}{3} + a_{00}\right) = \frac{1}{2}\left(\frac{2^{2n-1} - 2}{3} + a_{00}\right).
\end{aligned}$$

We now seek a recursion expressing $a_{00}(n+1)$ in terms of $a_{00}(n)$. A number of $2(n+1)$ digits (all possibly 0) can be formed from a number of $2n$ digits by appending either 00, 01, 10, or 11 to a number of $2n$ digits. Since these operations respectively increase the mod-3 remainder by 0, 1, 2, and 0 and the digit sum by 0, 1, 1, 2, we will obtain a number in the category $a_{00}(n+1)$ iff our initial number belonged to the category a_{00}, a_{21}, a_{11}, a_{00}. We can now build the recursion:

$$\begin{aligned}
a_{00}(n+1) &= a_{00}(n) + a_{21}(n) + a_{11}(n) + a_{00}(n) \\
&= 2a_{00}(n) + 2a_{11}(n) \\
&= 2a_{00}(n) + \frac{2^{2n-1} - 2}{3} + a_{00}(n) \\
&= 3a_{00}(n) + \frac{2^{2n-1} - 2}{3},
\end{aligned}$$

which we write as

$$a_{00}(n+1) - 3a_{00}(n) = \frac{1}{6} \cdot 4^n - \frac{2}{3}. \tag{2}$$

We now observe that the function $b_n = 4^n$ satisfies $b_{n+1} - 3b_n = 4^n$, so multiplying by $1/6$ achieves the first term on the right-hand side of (2); while the function $c_n = 1$ satisfies $c_{n+1} - 3c_n = -2$, so multiplying by $1/3$ achieves the second term on the right-hand side of (2). Thus the function

$$a_n = \frac{1}{6} \cdot 4^n + 1/3 = \frac{2^{2n-1}+1}{3}$$

satisfies (2), but it is not the desired $a_{00}(n)$ due to initial conditions: $a_1 = 1$ while $a_{00}(1) = 2$ (the relevant numbers being 00 and 11). So we employ the function $d_n = 3^n$, which satisfies $d_{n+1} - 3d_n = 0$ and $d_1 = 3$, which must therefore be divided by 3 to yield the required function:

$$a_{00} = \frac{2^{2n-1}+1}{3} + 3^{n-1}.$$

Solution 2. Let $d_{2n-1}, d_{2n-2}, \ldots, d_1, d_0$ be the binary digits of k from left to right (here we zero-pad so that k has $2n$ digits). The second and third conditions are respectively equivalent to

$$0 \equiv \sum_i 2^i d_i \equiv \sum_i (-1)^i d_i \mod 3 \quad \text{and} \quad 0 \equiv \sum_i d_i \equiv \sum_i (-1)^i d_i \mod 2.$$

So the conditions can together be written as

$$0 \equiv \sum_i (-1)^i d_i \mod 6. \tag{3}$$

Let $e_i = d_i$ when i is even, and let $e_i = 1 - d_i$ when i is odd. Then the e_i's, like the d_i's, range over all $2n$-tuples of 0's and 1's. Then (3) can be written as

$$\sum_i e_i \equiv n \mod 6. \tag{4}$$

The number of solutions to (4) is of course

$$a_n = \sum_{k \equiv n \bmod 6} \binom{n}{k}.$$

This is a sum of binomial coefficients multiplied by coefficients which are periodic mod 6; we evaluate it by applying the Binomial Theorem to sixth roots of unity. Let $\epsilon = e^{\pi\sqrt{-1}/3} = (1+\sqrt{-3})/2$ be one such, and note that

$$\sum_{r=0}^{5} \epsilon^{rk} = \begin{cases} 6 & \text{if } 6 \mid k \\ 0 & \text{if } 6 \nmid k \end{cases}$$

so

$$\begin{aligned}
a_n &= \sum_{k \equiv n \bmod 6} \binom{n}{k} \\
&= \frac{1}{6} \sum_{k=0}^{2n} \sum_{r=0}^{5} \epsilon^{r(k-n)} \binom{n}{k} \\
&= \frac{1}{6} \sum_{r=0}^{5} \left(\epsilon^{-rn} \sum_{k=0}^{2n} \epsilon^{rk} \binom{n}{k} \right) \\
&= \frac{1}{6} \sum_{r=0}^{5} \left(\epsilon^{-rn} (1+\epsilon^r)^{2n} \right) \quad \text{(Binomial Theorem!)} \\
&= \frac{1}{6} \sum_{r=0}^{5} \left(\frac{(1+\epsilon^r)^2}{\epsilon^r} \right)^n.
\end{aligned}$$

We use the simplification

$$\frac{(1+\epsilon^r)^2}{\epsilon^r} = (1+\epsilon^r)(1+\epsilon^{-r}) = (1+\epsilon^r)\overline{(1+\epsilon^r)} = |1+\epsilon^r|^2$$

to evaluate each of the six terms:

$$\begin{aligned} a_n &= \frac{1}{6}\sum_{r=0}^{5} |1+\epsilon^r|^{2n} \\ &= \frac{1}{6}(|1+1|^{2n} + |1+\epsilon|^{2n} + |1+\epsilon^2|^{2n} + |1-1|^{2n} + |1-\epsilon|^{2n} + |1-\epsilon^2|^{2n}) \\ &= \frac{1}{6}(4^n + 3^n + 1^n + 0^n + 1^n + 3^n) \\ &= \frac{1}{6}(4^n + 2\cdot 3^n + 2), \end{aligned}$$

which agrees with the previous answer.

7. Let x and y be real numbers, and define a sequence $a_0, a_1, a_2, \ldots$ by

$$a_n = \sum_{k=0}^{n} x^k y^{n-k}.$$

Suppose that a_m, a_{m+1}, a_{m+2}, a_{m+3} are integers for some $m \geq 0$. Prove that a_n is an integer for all $n \geq 0$.

Solution. By cancellation of terms we see that

$$a_{n+1} - xa_n = y^{n+1} \quad \text{and} \quad a_{n+1} - ya_n = x^{n+1}. \tag{5}$$

In particular, $a_{n+2} - xa_{n+1} = y^{n+2} = y \cdot y^{n+1} = y(a_{n+1} - xa_n)$, which we can write as

$$a_{n+2} = (x+y)a_{n+1} - xya_n. \tag{6}$$

We let $s = x+y$ and $t = xy$. Then the four given integral values of a_n yield a pair of linear equations in s and t (or, to be precise, s and $-t$):

$$\begin{aligned} a_{m+2} &= sa_{m+1} - ta_m \\ a_{m+3} &= sa_{m+2} - ta_{m+1} \end{aligned} \tag{7}$$

If the determinant $a_{m+1}^2 - a_{m+2}a_m$ is nonzero, these equations have a unique solution. In fact,

$$\begin{aligned} a_{m+1}^2 - a_{m+2}a_m &= a_{m+1}(a_{m+1} - xa_m) - a_m(a_{m+2} - xa_{m+1}) \\ &= a_{m+1}y^{m+1} - a_m y^{m+2} = y^{m+1}(a_{m+1} - ya_m) = x^{m+1}y^{m+1} = t^{m+1}. \end{aligned}$$

So we distinguish two cases.

Case 1. $t = 0$. Then without loss of generality $y = 0$, so $a_n = x^n$. The conclusion follows from the following lemma:

Lemma 1. *If x is a real number and n a nonnegative integer such that x^n and x^{n+1} are integers, then x is an integer.*

Proof. Note that $x = x^{n+1}/x^n$ is rational (if $x = 0$, the conclusion is immediate). Write $x = p/q$ in lowest terms with $q > 0$; then $x^n = p^n/q^n$ is also in lowest terms, hence $q = 1$. □

Case 2. $t \neq 0$. Then t^{m+1} is an integer, and likewise $t^{m+2} = a_{m+2}^2 - a_{m+3}a_{m+1}$ is an integer. So by Lemma 1 again, t is an integer. Now s is rational by Cramer's rule applied to (7). If we can prove s is an integer, then since $a_0 = 1$ and $a_1 = s$, we will be done by (6).

Write $s = u/v$ in lowest terms and assume that $v > 1$. Every a_n is rational; write $a_n = u_n/v_n$ in lowest terms. We claim that $v_n = v^n$ for every $n \geq 0$. The cases $n = 0$ and $n = 1$ are clear. We now induct, using (6):

$$\frac{u_{n+2}}{v_{n+2}} = \frac{u}{v} \cdot \frac{u_{n+1}}{v^{n+1}} - t \cdot \frac{u_n}{v^n} = \frac{uu_{n+1} - tv^2u_n}{v^{n+2}}.$$

Since u and u_{n+1} are coprime to v, the last fraction is reduced. So $v_{n+2} = v^{n+2}$ as desired. Hence $a_n \notin \mathbb{Z}$ for $n \geq 1$, which is a contradiction.

Remark. Many of the statements in the above proof can be proved more simply using the formula $a_n = \frac{x^{n+1}-y^{n+1}}{x-y}$ for $x \neq y$. The arguments above have been selected to avoid separately considering the case $x = y$.