Berkeley Math Circle
Monthly Contest 3
Due December 3, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

BMC Monthly Contest 3, Problem 3
Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

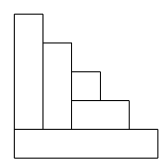
Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Fifty counters are on a table. Two players alternate taking away 1, 2, 3, 4, or 5 of them. Whoever picks up the last counter is the loser. Who has a winning strategy, the first player or the second?
2. How many divisors does 2013^{13} have? (As usual, we count 1 and 2013^{13} itself as divisors, but not negative integers.)
3. Define an *n-staircase* to be the union of all squares of an $n \times n$ grid lying on or below its main diagonal. The diagram shows a 5-staircase divided into 5 rectangles, each having a side of length 1. How many ways are there to divide a 10-staircase into 10 rectangles, each having a side of length 1? (Dissections obtained by reflecting the staircase are considered different.)



4. Let x, y, and z be real numbers such that $xyz = 1$. Prove that
$$x^2 + y^2 + z^2 \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$
5. Let $BCED$ be a cyclic quadrilateral. Rays CB and ED meet at A. The line through D parallel to BC meets ω at $F \neq D$, and segment AF meets ω at $T \neq F$. Lines ET and BC meet at M. Let K be the midpoint of BC, and let N be the reflection of A about M. Prove that points D, N, K, E lie on a circle.
6. For each $n \geq 1$, determine (in closed form) the number of integers k such that
   - $0 \leq k < 4^n$;
   - k is a multiple of 3;
   - The sum of the binary digits of k is even.
7. Let x and y be real numbers, and define a sequence $a_0, a_1, a_2, \ldots$ by
$$a_n = \sum_{k=0}^{n} x^k y^{n-k}.$$
Suppose that a_m, a_{m+1}, a_{m+2}, a_{m+3} are integers for some $m \geq 0$. Prove that a_n is an integer for all $n \geq 0$.