Berkeley Math Circle
Monthly Contest 1 – Solutions

1. Several ones are written in a row. It is permitted to insert one $+$ or $-$ sign between any two of them or leave the space blank. For example, using six ones one can achieve

$$1 \quad 1 \quad 1+1-1 \quad 1 = 101.$$

Is it possible to achieve the result 2013 using

(a) twenty ones and eight signs;

(b) twenty ones and nine signs?

Solution. (a) The answer is *yes*. Here is a solution:

$$1111 + 1111 - 111 - 111 + 11 + 1 + 1 + 1 - 1 = 2013.$$

(b) The answer is *no*. If there are nine signs, then there are 10 numbers being combined, and each of them is odd (since they end in 1). If we combine ten odd numbers, using addition and/or subtraction, we get an even number and therefore cannot get 2013.

2. Is it possible to paint each cell of a 4×4 table with one of 8 colors so that for every pair of colors, there are two cells painted with these colors having a common side?

Solution. The answer is *no*. Given 8 colors, the number of pairs of colors is $7+6+5+4+3+2+1 = 28$, and each pair of colors must appear on a pair of adjacent cells. We now count the pairs of adjacent cells. In each of the 4 rows there are 3 pairs of adjacent cells, yielding 12 horizontal pairs, and similarly there are 12 vertical pairs. Thus there are 24 pairs in all, not enough to accommodate the 28 pairs of colors.

3. Ten men sit side by side at a long table, all facing the same direction. Each of them is either a knight (and always tells the truth) or a knave (and always lies). Each of the people announces: "There are more knaves on my left than knights on my right." How many knaves are in the line?

Solution. Number the people from 1 to 10, from left to right according to their own perspective.

Person 1 is certainly lying, since he has no one to his left, and hence is a knave.

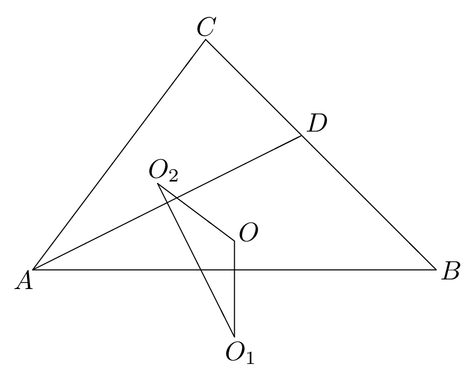
Person 10 is therefore a knight, since he has no one to his right and at least one knave (person 1) to his left.

Person 2 has exactly one knave to his left and at least one knight to his right, so he is a knave.

Person 9 has exactly one knight to his right and at least two knaves to his left, so he is a knight.

Continuing in this way, we find that person 3 is a knave, 8 is a knight, 4 is a knave, 7 is a knight, 5 is a knave, and 6 is a knight. Thus there are five knaves in the line.

4. Let ABC be a triangle, and let the bisector of $\angle BAC$ meet BC at D. Let O, O_1, O_2 be the circumcenters of triangles ABC, ABD, and ADC, respectively. Prove that $OO_1 = OO_2$.



Solution. We use the standard fact that the circumcenter of a triangle is the intersection of the perpendicular bisectors of the three sides. Note that $OO_1 \perp AB$ since both O and O_1 lie on the perpendicular bisector of AB. Similarly, $OO_2 \perp AC$ and $O_1O_2 \perp AD$. Hence the (counterclockwise) angle of rotation between lines OO_1 and O_1O_2 is the same as that between the perpendicular lines AB and AD; thus $\angle BAD = \angle OO_1O_2$, and similarly $\angle DAC = \angle O_1O_2O$. We are given that $\angle BAD = \angle DAC$; thus $\angle OO_1O_2 = \angle O_1O_2O$, that is, $\triangle OO_1O_2$ is isosceles with $OO_1 = OO_2$.

5. Determine, with proof, whether there is a function $f(x, y)$ of two positive integers, taking positive integer values, such that

- For each fixed x, $f(x, y)$ is a polynomial function of y;
- For each fixed y, $f(x, y)$ is a polynomial function of x;
- However, $f(x, y)$ does not equal any polynomial function of x and y.

Solution. The answer is yes. Consider the following expression:

$$f(x,y) = 1 + (x-1)(y-1) + (x-1)(y-1)(x-2)(y-2) + (x-1)(y-1)(x-2)(y-2)(x-3)(y-3) + \cdots.$$

Here, although the sum appears to be infinite, if we fix a value $y = y_0$, all but the first y_0 terms contain the factor $(y - y_0)$ and therefore equal 0. Therefore $f(x, y_0)$ is defined and indeed is a polynomial in x. (The initial term 1 is merely to ensure that $f(x, y)$ is always positive.) Symmetrically, when x is fixed, $f(x, y)$ becomes a polynomial in y.

It remains to prove that f is not a polynomial in x and y. If so, we can expand f as a finite sum of terms $cx^a y^b$; let a_0 be the largest exponent a occurring. Then, for every y_0, $f(x, y_0)$ is a polynomial in x of degree at most a_0. However, we see that $f(x, y_0)$ is a polynomial in x of degree $y_0 - 1$. Taking $y_0 = a_0 + 2$ yields a contradiction.

Remark. Another such function is $f(x,y) = \binom{x+y}{x} = (x+1)(x+2)\cdots(x+y)/y! = (y+1)(y+2)\cdots(y+x)/x!$.

6. On the coordinate line, the points with coordinates $0, 1, 2, \ldots, 2n-1$ are marked, where n is a positive integer. A flea starts jumping from the point with coordinate 0 and after $2n$ jumps returns there, having visited all the marked points. It is known that the total length of all jumps except the last one is $n(2n-1)$. Find the length of the last jump.

Solution. We claim that the length of the path formed by the first $2n-1$ jumps cannot exceed $n(2n-1)$. To prove this, take a path of maximal length (which exists because there are only finitely many paths the flea can take). Note that there cannot be three successive stations a_1, a_2, a_3 on the path with monotonic order ($a_1 < a_2 < a_3$ or $a_1 > a_2 > a_3$, since then the path could be lengthened by jumping directly from a_1 to a_3 and postponing the trip to a_2 to the end of the path. Therefore the path consists of a zigzag sequence of marked points

$$0 < r_1 > \ell_1 < r_2 > \ell_2 < \cdots < r_{n-1} > \ell_{n-1} < r_n,$$

and its length is

$$\begin{aligned}&(r_1 - 0) + (r_1 - \ell_1) + (r_2 - \ell_1) + \cdots + (r_{n-1} - \ell_{n-1}) + (r_n - \ell_{n-1})\\ &= 2(r_1 + \cdots + r_{n-1}) + r_n - 2(\ell_1 + \cdots + \ell_{n-1}).\end{aligned}$$

Writing this as

$$(r_1 + \cdots + r_{n-1}) + (r_1 + \cdots + r_n) - 2(\ell_1 + \cdots + \ell_{n-1})$$

makes apparent that it is at most

$$\begin{aligned}&((2n-1) + \cdots + (n+1)) + ((2n-1) + \cdots + n) - 2(1 + \cdots + (n-1))\\ &= (\underbrace{(2n-1) - (n-1)}_{n}) + \cdots + (\underbrace{n+1-1}_{n}) + (\underbrace{(2n-1) - (n-1)}_{n} + \cdots + (\underbrace{n+1-1}_{n}) + n\\ &= n(2n-1).\end{aligned}$$

In order for the path to achieve a length of $n(2n-1)$, all our inequalities must be equalities. In particular, the equalities

$$r_1 + \cdots + r_{n-1} = (2n-1) + \cdots + (n+1) \quad \text{and} \quad r_1 + \cdots + r_n = (2n-1) + \cdots + n$$

yield $r_n = n$. Hence the final jump from r_n back to 0 has length n.

7. Let a, b, c be real numbers between 0 and 1 inclusive. Prove that

$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} + a + b + c \le 3 + \frac{1}{3}(ab + bc + ca)$$

Solution. We will prove the following statement:

$$\frac{1}{1+a+b} \le 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

Note that summing this inequality with its two cyclically symmetric counterparts yields the desired result.

Clearing denominators, it suffices to prove that

$$\begin{aligned}(1+a+b)(6 - 3(a+b) + 2ab) &\ge 6\\ 2a^2b + 2ab^2 - 3a^2 - 3b^2 - 4ab + 3a + 3b &\ge 0\\ 2a(1-a)(1-b) + 2b(1-a)(1-b) + a(1-a) + b(1-b) &\ge 0,\end{aligned}$$

which is evident since $0 \le a, b \le 1$.