Center of Mass and Moment of Inertia in Plane Geometry

1. Center of Mass

We consider *systems of point masses (or charges) in the plane* $V = \{(P_1, m_1), \dots, (P_n, m_n)\}$ where $P_1, \dots, P_n$ are points in the plane and the $m_1, \dots, m_n$ are real numbers (most of the time positive) called weights or charges. We fix a point O, called the *origin*.

Definition 1. The *total mass* of a system V of point masses as above is the number $M = m_1 + \dots + m_n$ (sum of the weights) and (if $M \neq 0$) the *center of mass* of V is the point P for which

$$\overrightarrow{OP} = \frac{m_1 \cdot \overrightarrow{OP_1} + \dots + m_n \cdot \overrightarrow{OP_n}}{M}.$$

Properties. (a) The center of mass does not depend on the choice of the origin O.
(b) It is the only point P for which $m_1 \cdot \overrightarrow{PP_1} + \dots + m_n \cdot \overrightarrow{PP_n} = 0$.
(c) Calculating the center of mass, we can replace any subsystem by its total mass located in its center of mass. **This is the key property!**
(d) In particular, given three masses (A, a), (B, b), (C, c), let Z be the center of mass of $\{(A, a), (B, b)\}$. Then the center of mass of the three masses is the center of mass of $\{(Z, a + b), (C, c)\}$, therefore it lies on the segment CZ.
(e) If P is the center of mass of (A, a), (B, b), then $\frac{|PA|}{|PB|} = \frac{b}{a}$.

Points in a triangle. Given a triangle ABC with side lengths a, b, c and angles α, β, γ:
(a) The *centroid* of ABC is the center of mass of $\{(A, 1), (B, 1), (C, 1)\}$. *Corollary:* the centroid divides the medians in ratio 2 : 1.
(b) The *center of the inscribed circle* is the center of mass of $\{(A, a), (B, b), (C, c)\}$.
(c) The *orthocenter* is the center of mass of $\{(A, \tan\alpha), (B, \tan\beta), (C, \tan\gamma)\}$.
(d) The *circumcenter* is the center of mass of $\{(A, \sin 2\alpha), (B, \sin 2\beta), (C, \sin 2\gamma)\}$.

Problem 1 (Ceva's Theorem)**.** Given a triangle ABC and points $X \in BC$, $Y \in CA$, $Z \in AB$, prove that AX, BY, CZ are concurrent if and only if

$$\frac{|AZ|}{|ZB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} = 1.$$

Problem 2 (van Aubel's Theorem)**.** Given a triangle ABC and points $X \in BC$, $Y \in CA$, $Z \in AB$ such that AX, BY and CZ intersect in one point M. Prove that $\frac{|AM|}{|MX|} = \frac{|AY|}{|YC|} + \frac{|AZ|}{|ZB|}$.

Problem 3. Given a triangle ABC and points $X \in BC$, $Y \in CA$, $Z \in AB$ such that AX, BY and CZ intersect in one point M. Prove that $\frac{|AM|}{|MX|} \cdot \frac{|BM|}{|MY|} \cdot \frac{|CM|}{|MZ|} \geq 8$.

Problem 4. Let H be the orthocenter of a triangle ABC inscribed in a circle with center O. Prove that the sum of areas of two of the triangles OHA, OHB, OHC equal the area of the remaining triangle. *Hint: use the fact that* O*,* H *and the centroid* M *are collinear.*

Problem 5. We put a coin on each square of a chessboard and play the following game. A move consists of choosing two coins in the same row or column within the distance of 2 squares and stacking them on the square between them. Is it possible to put all the coins in one spot this way?

Problem 6. Find the center of mass of the *perimeter* of a triangle ABC.

Problem 7. Let $ABCD$ be a convex quadrilateral of area 1. We divide each side into 5 segments of equal length and draw a grid dividing the quadrilateral into $5^2 = 25$ quadrilaterals. Find the area of the quadrilateral in the middle.

2. Moment of Inertia

Definition 2. The *moment of inertia* of a system of masses $V = \{(P_1, m_1), \dots, (P_n, m_n)\}$ with respect to a point ("axis") X is the number

$$I_X(V) = m_1 \cdot |XP_1|^2 + \ldots + m_n \cdot |XP_n|^2.$$

Theorem (Parallel Axis Theorem)**.** Suppose that the total mass M of V is nonzero. Let P be the center of mass of V. Then for any point X,

$$I_X(V) = I_P(X) + M \cdot |PX|^2.$$

In particular, the moment of inertia of V with respect to X is minimal when X is the center of mass.

Problem 8. Given a triangle ABC, find the point X for which the sum $|AX|^2 + |BX|^2 + |CX|^2$ is the smallest.

Problem 9. Point P lies on the circumscribed circle of an equilateral triangle ABC. Prove that the sum

$$|AP|^2 + |BP|^2 + |CP|^2$$

is independent of the choice of P.

Problem 10. Point P lies on the circle inscribed in a triangle ABC. Prove that the sum

$$|AP|^2 \cdot |BC| + |BP|^2 \cdot |AC| + |CP|^2 \cdot |AB|$$

is independent of the choice of P.

Problem 11. Let M be the centroid of a triangle ABC. Prove that

$$|AB|^2 + |BC|^2 + |CA|^2 = 3(|MA|^2 + |MB|^2 + |MC|^2).$$

Problem 12. Is it possible to play the game from Problem 5 indefinitely?