Mathematical Magic for Muggles — April 16, 2013

Paul Zeitz, zeitzp@usfca.edu

Here are several easy-to-perform feats that suggest supernatural powers such as telepathy, "seeing fingers," predicting the future, photographic memory, etc. Each trick uses simple mathematical ideas that allow information to flow effortlessly and sneakily, among them

- simple, efficient "coding"
- parity and other invariants
- symmetry
- probability

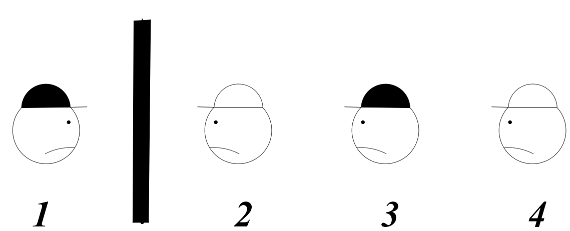
One can approach these activities in many ways. At first, you may want to figure out HOW to do a trick. Then, you want to know WHY it works. Finally, you should strive to understand REALLY WHY it works: is there a simple theme or principle behind your possibly complex explanation? Look for simple and general guiding principles. If you REALLY understand a trick, you should be able to create new tricks of your own!

Mostly about Coding and Communication

1 *Warm-up: Fingers That Can See*. The Magician deals cards on a table (not in a pile), placing them face up or face down on the command of the Participant, and stops dealing when the Participant says so.

Then the Magician is blindfolded. The Magician proceeds to put the cards into two piles, using his magical seeing fingers, so that, miraculously, each pile has exactly the same number of face-up cards!

2 *Warm-up: Buried in the Sand*. (Told to me by an 11-year-old girl.) Four unlucky people, labeled 1, 2, 3, 4, from left to right, are buried in sand up to their chins, as illustrated below.



There is a wall be separating #1 from #2; they are both facing this wall and that is all that they can see. However, #4 can see the back of the head of #3 and #2, and #3 can see the back of the head of #2. The evil person who imprisoned them put a hat on each person in such a way that no one can see the color of the hat that they are wearing, but can see the hats of people in front of them (unless there is a wall in the way). The evil person then said the following:

> I have put hats on each of you. Two hats are white, and two are black. If any of you can determine the color of your hat, and then explain your reasoning to me, I will free you all. If you say the wrong color, or if you merely guess correctly, and your reasoning isn't logical, then all of you will die.

Assume that all four people are intelligent and equally so. What happens? Explain.

3 *Zvonkin's Magic Table*. This trick is adapted from A. Zvonkin's book *Math From 3 to 7*, which I helped to translate and edit. Zvonkin ran a math circle for small kids in Moscow and entertained them by having them cover any four consecutive numbers in the table below (vertical or horizontal), and then he would instantly determine the sum! Was it a feat of memory? Telepathy?

5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0
7	1	7	2	3	7	1	7	2	3	7	1
2	7	6	1	4	2	7	6	1	4	2	7
5	6	1	6	2	5	6	1	6	2	5	6
5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0
7	1	7	2	3	7	1	7	2	3	7	1
2	7	6	1	4	2	7	6	1	4	2	7
5	6	1	6	2	5	6	1	6	2	5	6
5	6	1	6	2	5	6	1	6	2	5	6
1	0	5	5	9	1	0	5	5	9	1	0

4 *Concentration*. The Participant deals out cards in a 4×4 grid, putting some face up, some face down randomly. Then the Magician deals a few more cards, adding one more row and one more column. The Magician is then blindfolded and the Participant picks one card in the grid and turns it over (i.e., if it was face up, now it is face down, and if it was face down, now it is face up). The Magician takes off the blindfold and is miraculously able to spot the altered card. How?

5 *More Hats*. Ten people are now the victims of the Evil Villain from Problem #2. This is what E. V. tells them:

> In a few minutes, you will line up, and I will put hats on you. The hats will be black or white. You will be able to see the hat colors of the people in front of you, but you won't be able to see your hat or the hats of the people behind you. Starting at the back of the line, I will ask you to say your hat color. You are

> allowed to say the single word "black" or "white" just once, and otherwise, you are not allowed to communicate with each other at all.
>
> If you answer incorrectly, I will kill you instantly and loudly, so that the people in front of you will know. If you answer correctly, you will be spared.
>
> You can take a few minutes to confer on a strategy before I line you up. After that, you may not communicate with each other in any way except when you state your hat color.

What is the best strategy? How many people can be saved?

6 *Telepathic Teacher*. The Teacher, blindfolded, asks a Student to write a largish (four to six digits, say) number on the board. The student is instructed to then write the number backwards, and to subtract the smaller of the two from the larger, with other students quietly checking the work to make sure it is perfect.

Then the Teacher asks the student to circle one digit in the answer, and then say what the other digits are. The class is then asked to concentrate deeply on the circled digit. The Teacher is able, with high probability, to correctly name the digit.

How is this done? Why is it only with "high probability?"

7 *The Kruskal Count*. This telepathy trick can be done with cards or numbers. With cards, the Magician deals out an entire deck face up on a table, and asks the participant to mentally pick one of the first dozen or so cards and then use that card to tell him or her where to go next. If the card is an Ace, move one spot to the next card. If it's 2 through 9, go that many places. If it's a face card, move the number of letter of the card (i.e., Jack or King means move four, Queen means move five). Keep doing this until you can go no further. For example, if you start with the Jack of Hearts, you then move 4 cards down and perhaps that is an Ace of clubs. Then you move to the next card, the 7 of spades, and move 7 down, etc.

When the participant gets to the final card (the one where you cannot go further, because you'd go past the last card in the deck), he or she thinks hard about it. And the Magician manages to deduce the card.

The trick can also use a random list of numbers, or a semi-random one, such as the digits of π below.

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4
6	2	6	4	3	3	8	3	2	7	9	5	0	2	8	8	4	1	9	7
1	6	9	3	9	9	3	7	5	1	0	5	8	2	0	9	7	4	9	4
4	5	9	2	3	0	7	8	1	6	4	0	6	2	8	6	2	0	8	9
9	8	6	2	8	0	3	4	8	2	5	3	4	2	1	1	7	0	6	7

With a number table, the rule is simpler: Pick any starting point in the row, and move that many places, unless you hit 0, in which case you move one place. For example, if you start with the second digit (1), you move one place, to 4, then 4 more places, to 2, then 2 places, to 5, etc. Once again, the Participant mentally chooses a starting point, concentrates on the ending number, and the Magician magically guesses it!

Mathematical Card Tricks

Several of these tricks were researched, perfected, and classroom-tested this winter at the San Francisco Math Circle by SFSU grad students Jessica Delgado and Kelly Walker. I am indebted to them. In turn, they (and I) are also indebted to the recent *Magical Mathematics*, by Persi Diaconis and Ron Graham (Princeton University Press, 2012).

8 *Hummer Shuffle Tricks*. The three tricks below all employ the "Hummer Shuffle," which consists of picking up the first two cards of a deck, turning the two cards over, and replacing them on the top of the deck (i.e., card #1 becomes card #2 and card #2 becomes card #1, and both get turned over), followed by cutting the deck (you take the top n cards, where n is up to you, and lift them off the deck, then place them at the bottom, without turning the n cards over, so that now the top card is the previous $(n+1)$st and the bottom card is the previous nth card, etc. After doing a bunch of Hummer Shuffles, the cards in a deck are hopelessly messed up, since not only is the order permuted, but some of the cards will be face up and some will be face down. However, this shuffle is surprisingly orderly, as you will see.

(a) *Baby Hummer*. This trick only uses four cards. The Participant takes four cards, all facing the same way, and sneaks a peek at the bottom card. Then the Participant does the following:

1. Take the top card and place it on the bottom
2. Turn the current top card face up
3. Perform several Hummer Shuffles
4. Turn over the top card and put it on bottom
5. Put the current top card on the bottom without turning it over
6. Turn the top card over and leave it on top

Now spread the cards out and three cards will be facing one way and your original bottom card with be facing the other!

(b) *Nearly Perfect Mind Reading?* The Magician gives the Participant ten cards from A to 10, in order. The Participant then performs several Hummer Shuffles, thoroughly messing up the cards. The Magician is blindfolded. Then, the Participant starts reading off the cards in order, from the top of the disordered pile, telling the Magician what card it is. The Magician is able to guess whether the card is face up or face down, with nearly flawless accuracy (much better than 5 correct—the expected number due to random guessing)!

(c) *Royal Flush Hummer*. The Magician takes about half a deck and shows the cards in it to the Participant, who is invited to shuffle them. The magician then apparently messes the cards up further in a random way with respect to orientation (face-up vs. face-down). Then the Magician invites the Participant to continue messing up the cards with some Hummer-type shuffles. Then the Magician deals the cards into two piles, puts them together, and spreads them out. Exactly 5 cards are face-down. They miraculously form a royal flush!

9 *Scarne's Lie Speller*. The Magician spreads out cards from the top of a deck and invites the Participant to pick one. He does so, notes the card, and puts it on the top of the deck, without the Magician seeing it. The Magician then cuts the cards, and puts them behind her back. She says, "I will flip over a random card and put it in deck somewhere."

Then she spreads the cards out from the top, stopping at the one face-up card. She says that if this card is red, the Participant must tell the truth, but if it is black, then the Participant may lie. The Magician sets aside the cards that were above the face-up card, and holds the cards below the face-up card. She then asks the following questions:

1. "Is the card red or black?" Depending on the answer, she deals out that many cards (e.g., if the answer is "red," she deals "R-E-D" from her pile (the cards below the face-up card).
2. "Is the card seven, above seven, or below seven?" Again, she deals out the answer.
3. "Is the card hearts or diamonds/ clubs or spades?" (depending on the answer to #1).

When she deals the answer to this question, the chosen card magically appears!

10 *Random Numbers*. The Magician asks the Participant to choose a random number n between 1 and 20, and share this number with the audience without letting the Magician know. The Participant then removes the top n cards from the deck.

Next, the Magician deals 20 cards from the top of the diminished deck (which is missing n cards), and he asks the audience to notice the nth card dealt (without giving it away with body language!).

Next, an audience member is asked to estimate half the size of the now very diminished deck (it is missing $20+n$ cards). We call this number h. The Magician then deals h cards from the top, face-down. Then he places the stack of 20 cards on top of this, and puts the rest of the diminished deck on top of that (so the n cards removed at the start are still missing).

Finally, the Magician deals cards off the top, but at some miraculous point, stops, and it is the one that the audience noted!

11 *Guess from the Cut*. The Magician hands a deck of cards to several Participants, and asks one to cut the cards. Then she asks each of three Participants to draw from the top of the deck. She asks the first Participant to say the value of his card (not the suit), and asks the second to merely state his suit. For the third person, she only asks that the participant concentrate mentally on broadcasting his card. Of course, the Magician is able to correctly identify all three cards!

The Mysteries, Revealed!

1 *Fingers That Can See*. M watches and keeps track of the total number of face up cards. Call this number u. Then while blindfolded, M merely collects any u cards into a pile (making sure to keep their original orientation) and then flips this entire pile upside-down. Then this pile and the remaining cards have the same number of face-up cards. The reason: suppose that, among the u cards collected, that f of them are face up. Then $u-f$ are face down. However, in the pile of non-collected cards, $u-f$ must be face up (since the total number of face up cards is u). So flipping the chosen cards does what we want!

2 *Buried in the Sand*. Person #3 notes that #4 is *silent*, indicating that #3 has a black hat; for if her (#3's) hat were white, then #4 would see two white hats and immediately deduce that his (#4's) hat had to be black. So #3 says "black," after a short wait.

3 *Zvonkin's Magic Table*. The table is a repeating grid of 5×5 numbers arranged so that each row and each column sums to 20. Such grids are easy to make—try it yourself!—and now the trick is obvious: just look at the number adjacent to the covered area, and subtract this from 20.

4 *Concentration*. M arranges the cards in the extra row and column in such a way as to guarantee that each row and each column has an *odd* number of face-up cards. This is always possible, because there are an odd number of rows and an odd number of columns: Obviously, M can add a card to each of the four rows, and to each of the four columns, ensuring that these rows and these columns each have an odd number of face-up cards, but there is one card left to place, which will complete the new fifth row and fifth column. Place this card so that the new fifth row now has an odd number of face-up cards. At this point, each row has an odd number of face-up card, and since there are an odd number of rows, we know that the total number of face-up cards will be odd. Looking at columns, the first four columns each have an odd number of face-up cards, which totals to an even number. So the fifth column must have an odd number of face-up cards.

In any event, it is now easy to spot the altered card. Just look for the row with an *even* number of face-up cards, and the column with the same property. The altered card will be at this intersection.

5 *More Hats*. The prisoners agree that the person at the back of the line uses the following code: "white" means "the number of black hats that I see is even" and "black" means "the number of black hats that I see is odd." This person may or may not live, but allows the others to live, because now everyone can keep track, if they are careful. Even if someone makes a mistake, the people in front can recover.

6 *Telepathic Teacher*. This is a classic application of the "casting out nines" method which gives a nice algorithm for computing the remainder of a number when it is divided by 9: Just add the digits, repeating if needed. Why is this true? There are many possible explanations, many using algebra. But we can explain it without algebra, with an example. Suppose we want to know the remainder for 764 when divided by 9. We write

$$764 = 7\cdot 100+6\cdot 10+4$$

$$
\begin{aligned}
&= 7\cdot(99+1)+6\cdot(9+1)+4 \\
&= 7\cdot 99+7+6\cdot 9+6+4 \\
&= 7\cdot 99+6\cdot 9+(7+6+4),
\end{aligned}
$$

and clearly the first two terms of the sum above are multiples of 9, so the leftover (the sum of the digits) is the remainder.

Back to the trick. The original number has a certain remainder. The reversed number has the *same* remainder, since it has the same digit sum! So the difference will have a remainder of zero, i.e., will be a multiple of 9. Thus, when all but one of the digits in the difference are called out, the Teacher can add them up and figure out what is needed to make a multiple of 9. For example, if the Teacher hears, "8, 4, 0, and 7," the sum is (after adding digits when needed) 1, so the missing digit must be 8 (to get a sum of 9).

The reason that this trick only works with high probability, rather than certainty, is because the digit sum of the difference could end up being a multiple of nine! Then the missing digit is either 0 or 9. But in this case, the Teacher can just say, "I am seeing a round image. There is some interference. I cannot tell if there is a little hook at the bottom or not. So it is either 0 or 9." This will maintain the illusion of telepathy, albeit subject to interference.

7 *The Kruskal Count*. This trick works for the same reason that putting a hotel on Park Place is almost always a winning Monopoly strategy: eventually, someone will land at Park Place!

Pick the very first card (or digit) and plot out the evolution of this pick. Imagine, say, that each card (digit) that gets visited is colored green. Now consider a different starting point. This will engender a new sequence of visited locations. But observe that as soon as we reach a green location, we are locked into all the rest of the green locations.

So now, think of the green locations as "mines" or "Monopoly hotels that belong to our opponent." We start at some random point, and then our course is preordained (by the actual values of the cards or digits) but is also, in some sense, random. With digits, each step will be have length from 1 to 9, with each choice approximately equal (1 is more likely, since landing on 0 leads to a step size of 1). With cards, step sizes of 4 and 5 are somewhat more likely than the others, but otherwise, it's a random choice between 1 and 9.

In other words, each random sequence of digits (or shuffled deck of cards) plus a starting point yields a random sequence of step lengths, with approximately equal probabilities for each step length.

How do you avoid green locations? At each step, look for the nearest green location, and make sure not to step that distance. Just like Monopoly: if you are 8 steps from Park Place, you toss your dice, hoping not to get an 8. Since there are 9 possible step lengths, and only one bad one, at each turn, you have an $8/9$ probability of missing the next green location. Consequently, if you do this 15 times, the probability of missing *all* of the green locations is $(8/9)^{15}$, which is about 17%. Hence there is an 83% probability that you will hit a green spot and then get locked into the sequence that began with the very first location.

So that's how the Magician does the trick, by starting from the first spot and knowing that, with high probability, the Participant and the Magician will end up in the same place.

8 *Hummer Shuffle Tricks*. Consider a pile of cards, where some possibly are face up. Each card has a position (from #1, the top card, down to the last card), a value (where $A = 1$ and J, Q, K respectively equal 11, 12, 13), and an orientation (either face-up or face-down). All of these tricks depend on using an *even* number of cards and use one or both of the following lemmas.

> **Lemma 1**: Start with a pile of $2n$ cards, *all face-down*. After any number of Hummer Shuffles is performed, the number of odd-position cards that are face-up will equal the number of even-position cards that are face-up.

> **Lemma 2**: Start with a pile of $2n$ cards, *all face-down*, and *arranged in numerical order* (for example, $5, 6, 7, 8, 9, 10, J, Q$). then do any number of Hummer Shuffles. For each card, the sum of its position, value, and orientation (where we assign 1 to "face-up" and 0 to "face-down") will have the same parity.

For example, suppose the cards start with $5, 6, 7, 8$ from top to bottom, all face-down, and we turn over the first two and cut by taking the top card and putting on the bottom and then turn over the top two and cut by taking the top two cards and putting them on the bottom. Then we get, in order (using a bar to indicate "face-up"), from the starting position:

$$5, 6, 7, 8 \to \bar{6}, \bar{5}, 7, 8 \to \bar{5}, 7, 8, \bar{6} \to \bar{7}, 5, 8, \bar{6} \to 8, \bar{6}, \bar{7}, 5.$$

Now let's compute the sum of position plus value plus orientation for each card. The first card's sum is $1 + 8 + 0 = 9$. Card #2's sum is $2 + 6 + 1 = 9$. Card #3's is $3 + 7 + 1 = 11$, and the final sum is $4 + 5 + 0 = 9$. All of these are odd.

I leave it to the reader to prove these lemmas, but this should not be difficult. The harder part is thinking of the lemmas in the first place! We also leave it to the reader to use these lemmas (or other similar ideas) to explain (a).

Lemma 2 is used for (b), the Nearly Perfect Mind Reader trick. The Magician merely guesses the first answer, but of course the Participant will tell the Magician if he or she is correct or not. This establishes the parity of the sum, and the rest is (fairly) easy, but requires paying attention.

For (c), the Magician makes sure that there are an even number of cards in the pile, and that a royal flush is included among them. Then M cleverly arranges the orientation of the cards by examining successive pairs and flipping over the odd-positioned card ONLY if it belongs to the royal flush, and flipping over the even-positioned card ONLY if it doesn't belong to the royal flush. I am right handed, so I start looking at the cards from the right, so I use the mnemonic aid "**R**oyal flush cards get flipped if they are the **R**rightmost one in the pair."

At this point, some cards are face-up and some are face-down, but the following regularity has been imposed:

> *The odd-positioned royal cards have the same orientation as the even-positioned ordinary cards. Likewise, the even-positioned royal cards have the same orientation as the odd-positioned ordinary cards.*

Notice (VERIFY!) that Hummer Shuffling will not change this situation! So after a bunch of Hummer Shuffles (even ones where you flip over the top 4 cards, or any even number of cards), the Magician finally deals the cards out into two piles, alternating cards. M observes which pile has face-up royal cards, and takes this pile and surreptitiously turns it over and places it on the other pile. Now the only cards that are face down will be the royal cards!

9 *Scarne's Lie Speller*. This trick uses one bit of brain-dead deception and is otherwise completely straightforward. M starts out by making sure that the 14th card from the bottom of the deck is face-up and black.

When P picks a card and puts it on top, M cuts, making sure not to cut below the face-up card. Thus, after the cut, the chosen card is 14 cards below the face-up card. *We know exactly where it is!* The moral of the story:

> *Cuts don't do squat. But you already knew this. Your audience does not. They think that cutting the cards significantly messes up things.*

Then, when M says she is flipping a card behind her back, she actually does nothing (makes flipping noises, though). Next, she spreads the cards until a face-up one appears. Guess what? It's black.

Now M and P play a preordained guessing game, designed to merely deal out 14 cards, with the chosen one guaranteed to the the last one.

10 *Random Numbers*. The crux idea behind this trick is that $n+(-n)=0$. Keep it simple for a moment, and suppose that $h=0$. Then P takes n cards off the top of the deck, and M draws out 20 from the $(52-n)$-card deck, with the audience noting the nth one. Since $h=0$, M just puts the $(32-n)$-card deck on top of the 20-card deck. However, the audience's card is the nth from the top of the 20-card deck. Adding, we get $32-n+n=32$; thus M merely counts down to the 32nd card and this will be the target.

In the more general case, there will be h cards at the bottom, 20 cards in the middle (with the target card at the nth position from the top) and $32-n-h$ cards on top. So now M counts to the $32-h$th card. Easy!

11 *Guess from the Cut*. The key thing here is that M does not let anyone shuffle the cards, and remember: cuts do nothing to alter the cycle, merely changing the starting point. If you can "walk" the cycle, you're fine.

So there are many different ways to do this trick. All you need is a robust cycling of cards. I used the "Eight Kings" cycle, which uses the mnemonic aid "8 kings threatened to save 95 queens for a sick knave," which encodes

$$8, K, 3, 10, 2, 7, 9, 5, Q, 4, A, 6, J,$$

where we use the convention that $A=1$. Furthermore, we cycle suits with the pattern "CHaSeD" ($\clubsuit, \heartsuit, \spadesuit, \diamondsuit$). For example, the first card in the deck is the $8\clubsuit$, the next one is $K\heartsuit$, etc.

Then, if we know the value of one card and the suit of the next, we know everything!

12 *Mind-Reader?*

1. First, M and A establish a "lexigraphic" (dictionary) order for the cards in a deck. For example, the cards can be ordered "iTunes style" (first names have higher priority, so "John Smith" comes before "Zena Andrews," and hence "Three of Diamonds" comes before "Queen of Clubs"):[1]

$$A\clubsuit, A\diamondsuit, A\heartsuit, A\spadesuit, 2\clubsuit, \ldots, K\clubsuit, K\diamondsuit, K\heartsuit, K\spadesuit.$$

2. Within each suit, cards are ordered from 1 to 13, starting with Ace. Given two cards of the same suit, place them on a 13-hour clock. Notice that the no matter what the two cards are, one is at most 6 clockwise units away from the other. For example, the clockwise distance between $4\clubsuit$ and $K\clubsuit$ is 4, since you can travel 4 units clockwise from 13 to get to 4.

3. Next, A looks at the five cards. By the pigeonhole principle, there are at least two cards that are of the same suit. A choses one of these and puts it back in the deck and puts the other card on the top of the pile. A also computes the clockwise distance from this card to the "hidden" card. For example, suppose the five cards chosen are $3\clubsuit, J\heartsuit, 10\diamondsuit, Q\clubsuit, A\spadesuit$. A puts the $3\clubsuit$ back into the deck and puts the $Q\clubsuit$ on the top of the pile, and notes that the distance from $Q\clubsuit$ to $3\clubsuit$ is 4 (since $Q = 12$ and $12 + 4 \equiv 3 \pmod{13}$).

4. Now all A needs to do in encode a number between 1 and 6 with the remaining three cards in the pile. This can be done by agreeing in advance which of the $3! = 6$ lexigraphic permutations refers to which number between 1 and 6. For example, if we order permutations "alphabetically" like so:

$$(a,b,c),(a,c,b),(b,a,c),(b,c,a),(c,a,b),(c,b,a),$$

   Then we would encode "4" by placing the bottom three cards in the lexigraphic order (b,c,a). In our example, the three remaining cards are $J\heartsuit, 10\diamondsuit$ and $A\spadesuit$ and their lexigraphic order is

$$A\spadesuit < 10\diamondsuit < J\heartsuit.$$

   Hence the (b,c,a) ordering would correspond to $10\diamondsuit, J\heartsuit, A\spadesuit$.

5. So the final ordering (from top to bottom) for our example would be

$$Q\clubsuit, 10\diamondsuit, J\heartsuit, A\spadesuit.$$

[1] This method of ordering, rather than first ordering by suit, was suggested by one of my students, who noticed that often you don't need to even think about suits this way.