Spherical Geometry

Berkeley Math Circle, Sept 25, 2012

Consider the unit sphere in the space, i.e. all points at distance 1 from some center O.

Problem 1. What is a spherical line?

a. (line in the plane). Given three points A, B, C in the plane, use triangle inequality $|AC| \leq |AB| + |BC|$ to determine whether the point B lie on the segment AC.

b. For any points O, A, B, C in the space show $\angle AOC \leq \angle AOB + \angle BOC$.
Hint:
   - assume that the statement is false, i.e. $\angle AOC > \angle AOB + \angle BOC$;
   - choose a point B_1 inside the angle $\angle AOC$ so that $\angle AOB_1 = \angle AOB$, $|OB_1| = |OB|$;
   - draw a line segment with endpoints on the rays OA and OC through B_1;
   - use the following fact of plane geometry: if two triangles $\triangle BOC$ and $\triangle B'OC$ satisfy $OB = OB'$, $\angle BOC > \angle B'OC$ then $BC > B'C$.
   - Can you prove the fact above?

c. When the inequality in 1b is sharp?

d. Define the distance on the sphere as $|BC| = \angle BOC$. Does it coincide with the length of the shortest path on the sphere connecting B with C?

e. The *lines* (or *geodesic lines* or *geodesics*) are the shortest paths. Show that the lines in the sphere are *great circles* (a great circle is an intersection of the sphere with a plane passing through the center O of the sphere).

Problem 2. Symmetries of the sphere.

a. Show that the set of points equidistant from two given points on a sphere is a spherical line.

b. A *reflection* with respect to a line l is a distance preserving transformation that preserves the points of l pointwise and interchanges the hemispheres. Given two points A and B show that there exists a sequence of reflections which takes A to B. How many reflections do you need?

c. Given two congruent triangles T and T' find a sequence of reflections taking T to T'. How many reflections do you need?

d. Distance preserving transformations are called *isometries*. Show that any orientation preserving isometry of the sphere is a rotation of the sphere (with respect to a line passing through the center O in the space).

Problem 3. Polar correspondence.

Each great circle C corresponds to the pair of endpoints of the diameter DD' perpendicular to the plane containing C. We write $Pol(C) = D$ or $Pol(C) = D'$.

Each pair of antipodal points D and D' of the sphere corresponds to a great circle C lying in the plane perpendicular to DD'. We write $Pol(D) = Pol(D') = C$.

a. If a great circle C passes through a point A then the great circle $Pol(A)$ passes through the point $Pol(C)$.

b. The polar correspondence transforms points into lines, lines into points and the statement "a line l contains a point A" into the statement "the point $Pol(l)$ lies on the line $Pol(A)$".

   A *polar triangle* $A'B'C'$ for a spherical triangle ABC is defined as follows: to define A' consider two polar points to the line BC and choose one of them that lies on the same side with respect to BC as A does. B' and C' are defined similarly.

c. If triangle $A'B'C'$ is polar to ABC then ABC is polar to $A'B'C'$.

d. If α, β, γ are angles of the triangle ABC and a, b, c are its sides lenghs then the polar triangle $A'B'C'$ has angles $\pi - a, \pi - b, \pi - c$ and side lengths $(\pi - \alpha), (\pi - \beta), (\pi - \gamma)$. ($\pi$ denotes the angle of size 180^o).

Problem 4. Spherical triangle.

a. For a spherical triangle with angles α, β, γ show $\pi < \alpha + \beta + \gamma < 3\pi$.

b. Show that bisectors of angles in a spherical triangle pass through one point.

c. Show that the equidistant lines for the pairs of points (A, B), (B, C) and (C, A) pass through one point.

d. Show that the usual criteria of congruence of triangles work for spherical triangle.

e. Show an additional criteria of congruent triangles: two triangles are equal if their angles are equal. Hint: use the polars to obtain this criteria for free.

f. Show that there are no spherical rectangles.

Problem 5. Area of a spherical triangle.

A spherical *digon* is one of the four figure into which the sphere is partitioned by two spherical lines. Denote by $S(\alpha)$ the area of the digon with angle α.

a. Show that $S(\alpha) = 2\alpha$ (in a unit sphere).
   Hint: the area of the sphere of radius R is $4\pi R^2$.
b. Consider the lines AB, BC and CA on the sphere. How many spherical digons contaning triangle ABC can you count? How many digons contain the triangle $A_1B_1C_1$ antipodal to ABC? How many digons contain neither triangle ABC nor triangle $A_1B_1C_1$?
c. Show that $S(\triangle ABC) = \alpha + \beta + \gamma - \pi$.

Problem 6. Tilings of the sphere by triangles.

We say that a triangle T *tiles* the sphere, if there is a collection of triangles $T_1, \ldots, T_k$ congruent to T such that each point of the sphere belongs to at least one of T_i and if T_i and T_j have more than one point in common then they have a common side and T_i is symmetric to T_j with respect to the common side.

a. What can you say about the angles of triangles tiling the sphere?

   A *Coxeter triangle* is a triangle with angles π/k, π/l, π/m for integer $k, l, m \geq 2$.

b. Find all Coxeter triangles on the plane.
c. Find all Coxeter triangles on the sphere.
d. Show that the tiling of the sphere by triangles with angles $(\pi/2, \pi/3, \pi/3)$ corresponds to a regular tetrahedron.
e. Show that each regular polyhedron can be obtained from some triangular tiling.
f. Prove that there are exactly 5 regular polyhedra (and obtain them all from triangular tilings!). These polyhedra are called *Platonic solids*.

Problem 7. Miscellaneous.

a. For a spherical triangle with sides a, b, c show that $a + b + c < 2\pi$.

b. Find the angles of self-polar triangle (this is a triangle that coincides with its polar).

c. Prove that each spherical triangle has inscribed and circumscribed circles.

d. Let M and N be midpoints of AB and BC respectively. Show that $|MN| < |AC|/2$.

e. Given a spherical line segment of length α prove that the polars of all spherical lines intersecting this segment sweep out a domain of area 4α.

f. Given several spherical line segments whose sum of lengths is less than π prove that there exists a spherical line disjoint from each of the segments.

g. Let Π_1 and Π_2 be two parallel planes intersecting the sphere by circles C_1 and C_2. A student claimed that the area of the part of the sphere contained between the circle C_1 and C_2 does not depend on these circles. The student decomposed the annulus into four quadrilaterals, then he decomposed each of the quadrilaterals into two triangles, and since the sum of angles of these eight triangles is constant, one concludes that the area of the annulus is constant. Is the conclusion correct?

h. The area of the annulus described in 7.g depends on the distance between the planes Π_1 and Π_2 only.

i. (Eating a spherical cake). Spherical Winnie the Pooh and spherical Piglet live on the sphere and have a triangular cake. They divide the cake in the following way: Winnie choose a point A in the cake and Piglet choose a line through A cutting the cake. Then Winnie gets the smaller part and Piglet gets the rest. Where should Winnie the Pooh put the point A to get as much as possible?