Pell's Equation

Berkeley Math Circle

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1. Find all integers n such that n is simultaneously a triangular number and a square number.

2. Find all Pythagorean triples (a, b, c) with $a^2 + b^2 = c^2$ such that $b - a = 1$.

3. Find all pairs in positive integers m, n such that

$$1 + 2 + \cdots + m = (m + 1) + (m + 2) + \cdots + n.$$

4. Find all triangular numbers that differ by one from a square number.

5. Find all positive integers a, b such that

$$\binom{a}{b} = \binom{a-1}{b+1}.$$

Basic Properties of Pell's equation and Quadratic Fields

6. Show that the equation $x^2 - dy^2 = -1$ has no solutions for $d = 3$ or 8.

7. Let d be a fixed positive non-square integer. Show that if there is a solution to $x^2 - dy^2 = 1$, there are infintely many. Similarly, if there is a solution to $x^2 - dy^2 = -1$, there are infinitely many.

8. Let d be a non-square integer. If $a, b \in \mathbb{Q}$, we can consider numbers of the form $a + b\sqrt{d}$. Show that if $a, b, a', b' \in \mathbb{Q}$ and $a + b\sqrt{d} = a' + b'\sqrt{d}$, then $a = a'$ and $b = b'$.

9. Show that any polynomial with rational coefficients satisfied by α is also satisfied by its conjugate α'.

10. *Dirichelet's Theorem.* Let α be an irrational number. We can approximate α by rational numbers. Show that there are infinitely many rational numbers $\frac{x}{y}$ such that

$$\left| \frac{x}{y} - \alpha \right| < \frac{1}{y^2}.$$

Hint: Use the pigeon-hole principle on numbers of the form $y\alpha$ for $0 \leq y \leq n$.

The Continued Fraction Method

11. How can you tell when one continued fraction is larger than another?

12. Show that the successive convergents of a number α are alternately smaller than and larger than α (unless α is rational, in which case the last convergent equals α).

13. *Main Theorem of Continued Fractions.* Suppose $\alpha = \langle a_0, a_1, a_2, \dots \rangle$. Let p_n/q_n be a the n-th partial convergent. Show that

$$\langle a_0, a_1, a_2, \dots, a_n, x \rangle = \frac{xp_n + p_{n-1}}{xq_n + q_{n-1}}.$$

14. Show that $p_{n+1}q_n - p_nq_{n+1} = (-1)^n$.

Problems

15. Find all solutions to $x^2 - dy^2 = \pm 1$ for $= 2, 3, 5, 6, 7$.

16. Show that a right triangle with one angle $\pi/3$ can be well-approximated by right triangles with rational side lengths.

    The next five problems characterize those numbers with eventually periodic continued fraction expansion.

17. We are interested in continued fractions that are purely periodic. Suppose that $\alpha = \langle\overline{a_0, a_1, \ldots, a_n}\rangle$ and $\beta = \langle\overline{a_n, a_{n-1}, \ldots, a_0}\rangle$. Then show that $\alpha'\beta = -1$.

18. Show that any purely periodic continued fraction is a quadratic number α such that $\alpha > 1$ and $-1 < \alpha' < 0$. Call such a number a reduced quadratic irrational.

19. Show that for any d there are only finitely many P and Q such that $\frac{P+\sqrt{d}}{Q}$ is a reduced quadratic irrational.

20. If $\alpha = \langle a_0, a_1, a_2, \ldots, a_n, \alpha_{n+1}\rangle$ show that if α is a reduced quadratic irrational, then so is α_{n+1}.

21. For any quadratic irrational α define α_i by $\alpha = \langle a_0, a_1, a_2, \ldots, a_{i-1}, \alpha_i\rangle$. Show that α_i is eventually a reduced quadratic irrational. This is the point at which the continued fraction expansion becomes periodic. Thus any quadratic irrational number has an eventually periodic continued fraction expansion.

    Challenges

22. Consider the general quadratic Diophantine equation

$$Ax^2 + Bxy + C + Dx + Ey + F = 0.$$

    Show that the solution of this equation reduces to Pell's equation and case analysis.

23. A deep theorem of Roth says that if α is an algebraic irrational number, then for any $\epsilon > 0$ the inequality

$$\left|\frac{x}{y} - \alpha\right| < \frac{1}{y^{2+\epsilon}}$$

    has only finitely many solutions $\frac{x}{y}$. In other words, in looking for good approximations of α, we don't get much better than what the pigeonhole principle tells us. Use this to show that $x^3 - 2y^3 = 1$ has only finitely many solutions.

24. When does ab divide $a^2 + b^2 + 1$? Hint: Analyze the equation $a^2 + b^2 + 1 = kab$ for different values of k.

    Subtle questions to ponder

25. What is the length of the period of the continued fraction expansion of $\sqrt{d}$?

26. How do we obtain all solutions to $x^2 - dy^2 = N$? (For small N such solutions must come from the continued fraction expansion of $\sqrt{d}$, while for large N the answer relies on algebraic number theory.)

27. For which d does $x^2 - dy^2 = -1$ have a solution?