Warm-up Problems

Berkeley Math Circle (Advanced)
Chris Overton
2013.2.12

Problem 0. How do the following problems relate?

Problem 1. This problem should be review material for you.
a) Out of five people, how many ways are there to select three of them? Express your answer using factorials.
("n factorial" is defined as n! = n * (n-1) * (n-2) * … * 1, so for example 4! = 4 * 3 * 2 * 1 = 24.)

b) Pascal's triangle – hopefully, you have seen this before! Start with a single 1 in the "zeroth row", and in row n, define each number as the sum of the two numbers diagonally above:

```
                1
            1       1
        1       2       1
    1       3       3       1
1       4       6       4       1
```

We number both rows and non-zero entries k in each row starting at zero, so for example $\binom{4}{2} = 6$ (read "four choose two".)

- What is the sum of all the entries in row n? Why?
- Explain why $\binom{n}{k} = \binom{n}{n-k}$
- What is the formula for $\binom{n}{k}$?

Problem 2. Give a definition in Cartesian coordinates of your favorite (three-dimensional) triangular pyramid Δ^3. Can you make simple definitions that conserve volume but that allow vertices to change?

Problem 3. Let C^3 be a three-dimensional cube with sides of length 1.
a) Can you fill the entire volume of C^3 with pyramids Δ^3 whose corners are also corners of C^3? If so, how many pyramids does this take?
b) What is the largest regular pyramid (i.e. a tetrahedron) you can fit in C^3?

Problem 4. What is a sensible definition of C^n (compatible with Problem 3) for other dimensions n? Work your way up from n = 0.

Problem 5. Can you find a general formula for the number of k-(dimensional)-faces of C^n? For example, 0-faces are vertices, 1-faces are line segments...) Start by making a table for n (in rows) and k (in columns) up to 3 and try to see how to extend it. How does this table compare with Pascal's triangle? Hint: work backwards from the right of rows.

Problem 6. What is the smallest diameter for a three-dimensional ball B^3 that contains all of C^3?

Problem 7. Can you generalize problems 2, 3 and 6 to higher dimensions?