1 Warm-up problems:

(i) Post office problem. The post office in the country of Mland charges postage for letters which depends on the size of the envelope. The post office defines the size of a rectangular envelope as its height plus its width. You notice that you can tilt a long but narrow envelope and fit it in a more "square-like" one, so you get an idea: maybe I can put one envelope into another and *reduce* my postage! Prove that unfortunately this never works – if one envelope fits into another the postage of the outer one is always bigger.

(ii) Prove 2-D Mahler's inequality: For positive a, b, c, d

$$\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}.$$

2 Adding planar figures.

We will start with the following problem:

Problem. *Given two segments AB and CD in the plane, and a segment PQ with endpoints P on AB and Q on CD, what are all the places where the midpoint of PQ can land?*

Solution: To start assume AB and CD are not parallel. For a moment, fix Q. Then as P moves on AB the midpoint M traces out the midline of QAB. If we now let Q move on CD the midline of QAB will sweep out a parallelogram with vertices at midpoints of segments AC, AD, BX and BD.

If AB and CD are parallel but on different lines, we get a trapezoid $ABCD$ and M moves on it's midline. We get a similar result when $ABCD$ are all collinear. The length of teh resulting segment is $(AB + CD)/2$.

We can make the following general definition.

Definition 2.1. *For two plane figures F and G we define their half-sum $F \otimes B$ to be the set of all midpoints of segments with endpoints on F and G.*

Example 1: If F, G are single points P and Q we get the midpoint of FG, whcih we can denote $P \otimes Q$. If F is a segment AB and G is a point - we get the segment of length $AB/2$ parallel to AB. If F and G are parallel segments then $F \otimes G$ is the midline of the trapezoid that F and G create. If F and G are non-parallel segments then $F \otimes G$ is the parallelogram with vertices $A \otimes C$ $A \otimes D$, $B \otimes C$ and $B \otimes D$.

Example 2: Half-sum of two rectangles with sides are parallel to the coordinate axes and of size $a \times b$ and $c \times d$ is a rectangle who's sides are also parallel to the coordinate axes and of seizes $(a+c)/2 \times (b+d)/2)$. In fact in coordinates, if P is in F and Q is in G then their x-coordinates run through segments of size a and c, so the x-coordinate of the midpoint M runs through a segment of length $(a + c)/2$ and similarly for the y-coordinate.

Another operation that we can do with figures is to scale them.

Definition 2.2. *Fix a point O in the plane and a number k. For a given figure F we can consider segments OP for all P in F. We can scale the segment OP by a factor of k, getting new point OP' on ray OP, with $OP' = kOP$. The set of these points makes a new figure which we will denote kF.*

The stretching we performed is sometimes called a *homothety* (or homothecy, or homogeneous dilation. The new figure kF is similar to the original one, and is stretched by a factor of k. Of course this construction depends on choice of O, but a diffrent choice of O gives the same figure, only translated.

Now we can explain why we called the operation $F \otimes G$ a half-sum. Indeed, the name suggests that $2(F \otimes G)$ should be a sum, somehow.

This is indeed the case.

Definition 2.3. *Fix a point O. Then for a point P in F and point Q in G we can take OP and OQ and add them to get a new point S (if you are unfamiliar with vectors, wthis just means take the fourth vertex of the parallelogram with vertices O, P and Q). As P and Q vary, S traces out some figure which we call the sum of F and G and denote $F + G$.*

We remark that just like changing O did not change the shape or orientation of kF but just translated it, parallel translating one of the figures F or G also change $F + G$ by parallel translation.

Of course the point at the tip of $\frac{1}{2}OS$ is M, the midpoint of P and Q, so we see that $1/2(F + G) = F \otimes G$, and so $F \otimes G$ is indeed a half-sum. This may make it easier to find the half-sums of figures.

Exercise: Find sums (or half-sums) of the following figures.

(i) Two non-parallel (to each other) rectangles.

(ii) Equilateral triangle and one of its sides.

(iii) Two triangles into which a diagonal divides a square.

(iv) A square and an equilateral triangle sharing a side.

(v) A disc and a segment.

(vi) Two circles of different radii.

(vii) A half-circle with itself.

(viii) Two half-circles that make up a circle

(ix) F is two adjacent sides of a regular pentagon (just the 2 segments), G is the other three sides.

We can now not only multiply figures by numbers, but also add them! This is pretty cool. The addition operation we introduced is called the Minkowski sum, after Hermann Minkowski who invented it.

For some of our applications we will need to understand what happens when we add a polygon to a disc.

Problem. *Suppose we have a convex polygon F We consider the figure $N = F + rD$ for a number r and the unit disc D. Show that N is the r-neighbourhood of F - theset of all points at distance at most r from F. Show that the area of N is $A(F) + A(rD) + rp = A(F) + rp + \pi r^2$, where $A(G)$ is the area of figure G, and p is the primeter of F.*

With this in mind you may want to go back to the post office problem and see if the above helps (Hint: Consider $I + rD$ and $O + rD$ for the inner envelope I and outer envelope O).

This addition operation has the following amazing property.

Theorem 2.4. *(Brunn-Minkowski inequality) Lets denote the area of a figure F by $A(F)$. Then*

$$\sqrt{A(F+G)} \geq \sqrt{A(F)} + \sqrt{A(G)}.$$

Proof. Let's start with the case when F and G are rectangles with sides parallel to the coordinate axes (which we will call "bricks")). If F and G are of size $a \times b$ and $c \times d$ then $F + G$ is also such a rectangle and is of size $(a + c) \times (b + d)$. The inequality we want is exactly the 2-D Mahler inequality from the warm-up, so it's true.

The idea now is to reduce the proof to the case of two figures made out bricks. Indeed if F_n and G_n are brick figures that are contained in F and G and approximate them better and better so that $A(F_n)$ approaches $A(F)$ and $A(G_n)$ approaches $A(G)$. Then since the F contains F_n and G contains G_n, we see that $F + G$ contains $F_n + G_n$ (why?). Hence $A(F + G) \geq A(F_n + G_n)$ and by Brunn-Minkowski for brick figures $\sqrt{A(F_n+G_n)} \geq \sqrt{A(F_n)} + \sqrt{A(G_n)}$ so $\sqrt{A(F+G)} \geq \sqrt{A(F_n)} + \sqrt{A(G_n)} \to \sqrt{A(F)} + \sqrt{A(G)}$.

What remains now is to prove the Brunn-Minkovski inequality for figures that are made out of collections of such rectangles . If both F and G are a single brick that is the case above. So either F or G has at least two bricks, and we can assume it's F (why?).

Then there is a vertical or horizontal line that l with some of the bricks of F one one side of l and some on the other. We will assume that it's horizontal (if it's vertical the argument is similar). By changing our coordinate system we may assume that the line is in fact $y = 0$. Let F^t be the part of F on top of l and F^b the part of F under l. We next translate G (which as you recall does not change the shape of $F + G$) so that it is divided by l in the same ratio as F, splitting it into two parts G^t and G^b. Thus we have $A(G^t)/A(G) = A(F^t/A(F) = r$

The figure $F^t + G^t$ is contained in the upper half-plane and the figure $G^b + F^b$ is in the lower half-plane. So the two pieces do not overlap, hence $\sqrt{A(F+G)} \geq \sqrt{A(F^t+G^t)} + \sqrt{A(F^b+G^b)}$. Now since F^t and G^t are a simpler pair than F and G (contain fewer bricks together) we can assume (by induction on total number of bricks) that the Brunn-Minkowski inequality is true for F^t and G^t, so that

$$A(F^t+G^t) \geq (\sqrt{A(F^t)} + \sqrt{A(G^t)})^2 = (\sqrt{rA(F)} + \sqrt{rA(G)})^2 = r(\sqrt{A(F)} + \sqrt{A(G)})^2$$

Similarly for F^b and G^b

$$A(F^b+G^b) \geq (\sqrt{A(F^b)}+\sqrt{A(G^b)})^2 = (\sqrt{(1-r)A(F)}+\sqrt{(1-r)A(G)})^2 = (1-r)(\sqrt{A(F)}+\sqrt{A(G)})^2$$

Adding these up we get the Brunn-Minkowski inequality for F and G, which completes the proof. □

3 Queen Dido problem.

We shall now apply the Brunn-Minkowski inequality to solve one of the most famous problems from antiquity.

According to Greek mythology as recorded among other places in Virgil's "Aeneid" Dido was a daughter of the king of Tyre (city in modern-day Lebanon) but was forced to leave and travel to North Africa. There she encountered king Iarbas and offered to buy some land from him. King Iarbas took her jewels for payment, but told her he would only give her as much land as could be encompassed by an ox hide. She agreed, but cut the ox hide into thin strips and made it long enough to surround an entire nearby hill, which was therefore afterwards named Byrsa "hide".

Put yourself in Dido's shoes for the moment. In which shape should you arrange your hide-strips if you want to get the most land?

This is known now as "Queen Dido problem":

Problem. *Of all possible closed curves in the plane of fixed length, find the one that encloses the maximal area.*

This is also known as the "isoperimetric problem", from Greek words for "equal" and "measure around".

This problem was solved by Jacob Steiner in 19th century, but the proof contained a flaw - Steiner had to assume that a figure of maximal area actually exists! This is intuitively obvious, but hard to prove rigorously. Luckily for us, we can solve the Queen Dido problem by using the Brunn-Minkowski inequality, and the proof is very short:

Indeed for any polygon P with perimeter p and area S we have by problem 1 and the Brunn-Minkowski inequality $\sqrt{A(P+D)} = \sqrt{S+p+\pi} \geq \sqrt{S} + \sqrt{\pi}$.

Squaring both sides $p \geq 2\sqrt{\pi S}$, or $S \leq p^2/4\pi = \pi(p/2\pi)^2$, which says that S is no bigger than the area of the disc of same perimeter!

We note that we can get the inequality for any convex P (not necessarily polygonal) by approximating it by polygons arbitrarily well (and for arbitrary P the problem can be reduced to the convex case).

We note that there are many more cool things related to Minkowski sum and Brunn-Minkowski inequality, but we will stop here.

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4 Some more problems.

(i) Prove that if F and G are convex, then $F + G$ and kF are convex.

(ii) Suppose for two figures $\sqrt{A(F+G)} = \sqrt{A(F)} + \sqrt{A(G)}$. What can we say about F and G? (hint: First solve this for two bricks; then look through the proof of the Brunn-Minkowski inequality. When do we have equality?).

(iii) Notice that when we defined kF or $F + G$ we never used that F and G are 2-dimensional. Go over the definitions and convince yourself that they work for 3-dimensional and, if you know about those, even n-dimensional figures.

(iv) Prove general Mahler's inequality: For positive x_k and y_k

$$\prod_{k=1}^{n}(x_k + y_k)^{1/n} \geq \prod_{k=1}^{n} x_k^{1/n} + \prod_{k=1}^{n} y_k^{1/n}.$$

(v) Let P be a 3-D polyhedron and $B^3(r)$ a 3-D ball of radius r. What is $P + B^3(r)$? What is the volume of $P + B^3(r)$?

(vi) The 3-D Brunn-Minkowski inequality says that $V(F + G)^{1/3} \geq V(F)^{1/3} + V(G)^{1/3}$, where $V(F)$ is the volume of the figure F.

Use the previous problems and the 3-D Brunn-Minkowski inequality to show that of a sphere of unit surface area volume at least as large as any polyhedron of unit surface area. (Warning this is slightly trickier than what we did in 2-D. Use small r.)

(vii) Airline luggage problem. The airlines all over the world limit the size of luggage by the sum of dimensions – height plus width plus depth. Inspired by your post office ideas, you think perhaps for you can put an oversize suitcase inside another suitcase which has smaller size. Prove that this too, would never work.

(viii) Prove that (an interior of a) convex polygon F is centrally symmetric if and only if it cam be written as a Minkowski sum of some number of segments.

(ix) Another version of the story of Queen Dido has her separate a piece of land along an ocean shore. Solve this version of the Queen Dido problem – that is, given a line l (the ocean shore), find a curve γ of fixed length p which together with l cuts off the largest area.

(x) Yet another version - suppose the endpoints of the curve are some pre-determined points A and B on the line l. What is the curve of given length p that together with AB cuts off the largest area?

(xi) Of all the curves with endpoints on the sides of a given 45 degree angle and fixed length p, find the one that cuts off the largest area off that angle.

(xii) Find the length of the smallest curve dividing an equilateral triangle into two parts of equal area.