Some problems to consider...

1. Suppose we have an infinitely big hotel with the rooms numbered $1, 2, 3, \ldots$. Each room is occupied. If a new guest arrives at the hotel, can you move the current occupants to make room for the new arrival?

2. In the same full hotel, an infinite number of new guests arrive: numbered $1, 2, 3, \ldots$. Can you make room for these guests?

3. What if we have an infinite number of buses numbered $1, 2, 3, \ldots$, each of which contain an infinite number of guests (again numbered within each bus)? Can we make room for them all? (Problems 1-3 are called *Hilbert's Hotel*)

4. Find a bijection between the natural numbers $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$.

5. Find a bijection between $\mathbb{N}$ and the integers $\mathbb{Z}$.

6. Find a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

7. Argue that there exists a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \times \cdots \mathbb{N}$, where the right side is some finite product of $\mathbb{N}$ with itself. *Hint: Use induction and the previous result Remark: Sets in bijection with* $\mathbb{N}$ *are called* countable.

8. Let S be the set of infinite binary sequences $a_1a_2a_3, \ldots$ where the a_i are 0 or 1. Show that there does not exist a bijection between $\mathbb{N}$ and S *Hint: use a proof by contradiction.*

9. Find a bijection between S and the set of real numbers in $[0, 1]$. (So $[0, 1]$ is not countable).

10. Show that if there exists an injection $f : A \to B$ and an injection from $g : B \to A$, then there exists a bijection between A and B. (This is called the *Cantor-Bernstein-Schroeder theorem*). *Hint: If* $x \in A$, *Consider 'chains' of the form* $x \mapsto f(a) \mapsto g(f(a)) \mapsto f(g(f(a))) \mapsto \cdots$. *Can these chains be continued forward indefinitely? Backward? What different things can happen?*

11. Using C-B-S and #6, argue that $\mathbb{Q}$ is countable.

12. The power set of A is the set of all subsets of A, denoted $P(A)$. Show that there cannot exist a bijection between A and $P(A)$. (*Hint: This is similar to* #8*)*. Remark: We now introduce the Axiom of Choice

13. *Show that there exists an injection* $f : A \to B$ *if and only if there exists a surjection* $g : B \to A$.

14. *Show that the Well-Ordering Principle implies the Axiom of Choice.*

15. *Show that Zorn's Lemma implies the Well-Ordering Principle. (*Hint: Consider all well-orderings of all subsets of X)

16. Show that the Axiom of Choice implies Zorn's Lemma. (*Hint: Suppose* $(X, \leq)$ *violates Zorn's Lemma. Then the Axiom of Choice implies there exists a function* g*, which takes a chain* $\mathcal{C}$ *of* X *to a chain* $\mathcal{C}'$ *of* X *which contains* $\mathcal{C}$ *and also contains an element bigger than any element in* $\mathcal{C}$*).*

17. *Show that if* A *and* B *are any two sets, their cardinalities can be compared.* Remark: Whether there is a cardinality between the cardinality of the integers and the cardinality of the reals is called the *Continuum Hypothesis.* It was shown to be independent from the usual axioms of set theory, including the Axiom of Choice.

18. *Show that, for any infinite set* X*,* $X \times \mathbb{N}$ *has the same cardinality as* X*.*

19. *Show that, for any infinite set* X*,* $X \times X$ *has the same cardinality as* X*.*