Combinatorics and Games

April 23, 2013

Problem 1.

In a forest each of 9 animals lives in its own cave, and there is eactly one separate path between any two of these caves. Before the election for the Forest Gump, King of the Forest, some of the animals make an election campaign. Each campaign-making animal – $\mathcal{FGC}$ (Forest Gump Candidate) – visits each of the other caves exactly once, uses only the paths for moving from cave to cave, never turns from one path to another between the caves, and returns to its own cave at the end of the campaign. It is also known that no path between two caves is used by more than one $\mathcal{FGC}$. Find the maximum possible number of $\mathcal{FGC}$'s.

Problem 2.

Let $a_1, a_2, \ldots, a_n$ be the first row of a triangular array with $a_i \in \{0, 1\}$. Fill in the second row b_1, $b_2, \ldots, b_{n-1}$ according to the rule $b_k = 1$ if $a_k \neq a_{k+1}$, $b_k = 0$ if $a_k = a_{k+1}$. Fill in the remaining rows similarly. Determine with proof the maximum possible number of 1's in the resulting array.

Problem 3.

Red and White are playing the game of hex. They alternate turns placing a stone of their color in an empty cell on a hexagonal $n \times n$ grid (see picture). Each player's goal is to form a connected path of their stones linking the opposite sides of the board marked by their color, before the opponent connects their sides in a similar fashion.

Prove that hex cannot end in a draw, in other words, if the board is filled in with red and white stones, there is either a white path connecting the opposite white sides or there is a red path connecting the opposite red sides.

Problem 4. (IMO 2011)

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. We are to place each of the n weights on the balance, one after the other, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all the weights have been placed.

Determine the number of ways in which this can be done.

Problem 5. (IMO 2000)

100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

Problem 6. (IMO 1997)

An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

(a). there is no silver matrix for $n = 1997$;
(b). silver matrices exist for infinitely many values of n.

Problem 7. (IMO 2006)

Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the bounary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.

Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

Problem 8. (IMO 2007)

In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In aprticular, any group of fewer than two competitors is a clique). The number of members is called its *size*.

Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.