Berkeley Math Circle
Monthly Contest 7
Due April 9, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 7
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A rectangle has area 1. Prove that its perimeter is greater than or equal to 4.

   Remark. Proofs based on assertions from calculus will not be accepted.

2. Twenty-one playing cards are arranged in a row, alternately face up and face down, with the two end cards face down. We are permitted to make two types of moves:

   - Switch the positions of two cards, keeping their orientations the same.
   - Flip over two adjacent cards.

   Is it possible to perform the moves in such a way that the playing cards end up alternately face up and face down with the two end cards face up?

3. Prove that the numbers
   $$9801, \quad 998001, \quad 99980001, \quad 9999800001, \quad \ldots$$
   are all perfect squares.

4. A point P lies inside a square $ABCD$. The angles PAB, PBA, PCD, and PDC are denoted by α, β, γ, and δ.

   (a) Show that if $\alpha = \beta = 60°$, then $\gamma = \delta = 15°$.

   (b) Show that if $\gamma = \delta = 15°$, then $\alpha = \beta = 60°$.

5. Let p be an odd prime. Determine the positive integers x and y ($x \leq y$) for which $\sqrt{2p} - \sqrt{x} - \sqrt{y}$ is positive and as small as possible.

6. Find all positive integers n such that the binomial coefficient $\binom{2n-1}{n-1}$ is odd.

7. A circle ω and a triangle ABC intersect in exactly six points. Prove that the incenter of $\triangle ABC$ lies inside ω.