Berkeley Math Circle
Monthly Contest 6
Due March 12, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 6
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

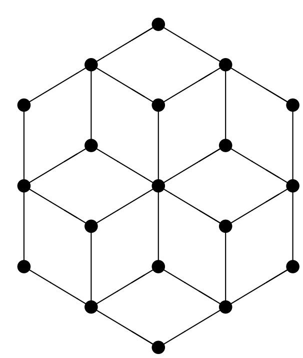
Problems

1. A $10 \times 10 \times 10$-inch wooden cube is painted red on the outside and then cut into its constituent 1-inch cubes. How many of these small cubes have at least one red face?

2. The whole numbers from 1 to 100 are each written on an index card, and the 100 cards shuffled in a hat. Twenty-six cards are drawn out of the hat at random. Prove that two of the numbers drawn have a difference of 1, 2, or 3.

   Remark. Notice that, despite the thorough mixing of the cards in the hat, this isn't a probability problem! The question asks not about the most likely outcomes but about all possible outcomes, and your solution should reflect this.

3. A science museum has the design shown, with 19 exhibits (shown as dots) connected by 30 hallways.

   (a) Is there a route that travels along each hallway exactly once? (Some exhibits may be visited multiple times.)

   (b) Is there a route that visits each exhibit exactly once? (Some hallways may remain unused.)

   Remark. Each part demands either an example of such a route (for a "yes" answer) or a proof that no such route exists (for a "no" answer).

4. Five consecutive vertices of a regular 2013-gon are given. Prove that one can reconstruct the entire 2013-gon using straightedge alone.

5. Determine whether there is a polynomial $f(x)$ such that

    - Every coefficient of f, from the leading coefficient down to the constant term, is either 1 or -1.
    - $(x-1)^{2013}$ evenly divides $f(x)$ (this means that their quotient is a polynomial).

6. Let $a_1, a_2, \ldots, a_{2013}$ be real numbers satisfying the following conditions:

    - $a_1 = a_{2013} = 0$;
    - $|a_i - a_{i+1}| < 1$, for $1 \leq i \leq 2012$.
    - $\sum_{i=1}^{2013} a_i = 0$.

    Find the greatest possible value of the sum $\sum_{i=1}^{m} a_i$, where m $(1 \leq m \leq 2013)$ is allowed to vary, in addition to the sequence $\{a_i\}$.

7. Find all positive integers n such that $n \mid 2^n - 1$.