Berkeley Math Circle
Monthly Contest 5 – Solutions

1. Calculate, with proof, the last digit of
$$3^{3^{3^{3^{3}}}}.$$

   Remark. Note that this means 3^(3^(3^(3^3))), not (((3^3)^3)^3)^3.

   Solution. When 3 is raised to the successive powers $1, 2, 3, 4, \ldots$, the units digits are $3, 9, 7, 1, \ldots$. From then on, since the digit 1 has been reached, the units digits will repeat in this cycle of four elements. So it is necessary to find the remainder when
$$n_4 = 3^{3^{3^{3}}}.$$
   is divided by 4.

   When powers of 3 are divided by 4, the remainders are $3, 1, \ldots$. Here the number 1 appears after two steps, and the remainders therefore repeat in a two-element cycle. So it is necessary to find the remainder when
$$n_3 = 3^{3^{3}}$$
   is divided by 2.

   But n_3 is clearly odd, so n_4 has a remainder of 3 when divided by 4 and the original number n_5 has a last digit of 7.

2. Prove that any triangle can be dissected into five isosceles triangles. (The pieces must not intersect except at their boundaries, and they must cover the given triangle.)

   Solution. We present one of many possible solutions.

   First, draw the height to the longest side (or one of the longest sides), as shown in Figure 1. This height will always lie inside the triangle and will divide it into two right triangles. We will cut one of these right triangles into two isosceles pieces and the other into three.

   For the former construction, we can simply join the midpoint of the hypotenuse to the opposite vertex, as shown in Figure 2. The two resulting triangles are isosceles by the familiar theorem that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

   We now turn to the problem of cutting a right triangle XYZ into three isosceles triangles. If the given triangle is *not* isosceles, the perpendicular bisector of the hypotenuse intersects the longer leg, labeled YZ in Figure 3, at a point W. Drawing XW cuts the triangle into an isosceles triangle XWY and a right triangle XWZ; the latter can be chopped up into two isosceles triangles by the preceding technique. If instead the given right triangle is isosceles, it can easily be divided into three right isosceles triangles by the method shown in Figure 4.

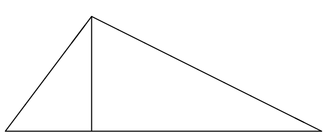

Figure 1

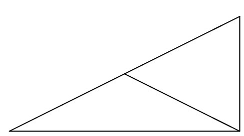

Figure 2

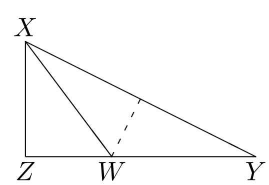

Figure 3

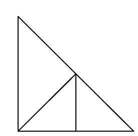

Figure 4

3. Alice and Bob play the following game on the whiteboard. First, Alice writes an odd number in binary on the board. Then, beginning with Bob, the players take turns modifying the number in one of two ways: subtracting 1 from it (preserving the binary notation), or erasing its last digit. When the whiteboard is blank, the last player to have played wins. Which player has a winning strategy?

   Solution. Bob can win using the following strategy: Play so as to leave an empty whiteboard or a number with an odd number of terminal zeros (preceded by a 1). We claim that Bob can fulfill this requirement at every move.

   On his first move, or indeed any move in which Alice leaves an odd number on the board, it is easy to see how the requirement can be fulfilled. Unless the number on the board is 1 (in which case Bob wins instantly), it ends in a 1 followed by either an even or an odd number of zeros; in these two cases Bob should subtract one and erase the 1 respectively.

   If Alice leaves an even number on the board, it must be by erasing a final 0 from a number left by Bob. Bob can then erase another zero, and the number of terminal zeros will remain odd.

4. Let $a \le b \le c \le d$ be real numbers such that

$$a + b + c + d = 0 \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 0.$$

Prove that $a + d = 0$.

Solution 1. Let us rearrange the first equation to read

$$a + b = -(c + d) \tag{1}$$

and the second to give

$$\frac{1}{a} + \frac{1}{b} = -\frac{1}{c} - \frac{1}{d}$$
$$\frac{a+b}{ab} = -\frac{c+d}{cd} \tag{2}$$

If $a + b \neq 0$, then we can divide (1) by (2) to yield $ab = cd$. So we conclude that

$$a + b = 0 \quad \text{or} \quad ab = cd.$$

In exactly the same manner, we derive that

$$a + c = 0 \quad \text{or} \quad ac = bd$$

and

$$a + d = 0 \quad \text{or} \quad ad = bc \quad \text{(this equation will not be needed).}$$

For the sake of contradiction, we are assuming that $a + d \neq 0$. Suppose that $a + c = 0$ holds; since $a + b + c + d = 0$, we get $b + d = 0$, whereas we know $a \le b$ and $c \le d$. We conclude that $a = b$ and $c = d$, whence $a + d = a + c = 0$. Thus $a + c \neq 0$, and for similar reasons $a + b \neq 0$.

Multiplying together the equations $ab = cd$ and $ac = bd$ and canceling the nonzero (why?) factor bc gives $a^2 = d^2$. So either $a = -d$, which is what we assumed impossible, or $a = d$, which yields the absurdity $a = b = c = d = 0$.

Solution 2. Consider the polynomial

$$f(x) = (x - a)(x - b)(x - c)(x - d)$$

with roots a, b, c, d. The cubic coefficient of f is $-(a + b + c + d) = 0$; the linear coefficient is

$$-(abc + abd + acd + bcd) = -abcd \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) = 0.$$

Consequently f has the form $f(x) = x^4 + px^2 + q$, and there are complex numbers u and v such that

$$f(x) = (x^2)^2 + px^2 + q = (x^2 - u)(x^2 - v) = (x + \sqrt{u})(x - \sqrt{u})(x - \sqrt{v})(x + \sqrt{v}).$$

Since f has only real roots, u and v must be real and nonnegative. WLOG $u \le v$; then $-\sqrt{v} \le -\sqrt{u} \le \sqrt{u} \le \sqrt{v}$ and we can identify the two factorizations of f:

$$a = -\sqrt{v}, \quad b = -\sqrt{u}, \quad c = \sqrt{u}, \quad d = \sqrt{v}.$$

Hence $a + d = 0$.

5. Let ω be a circle with diameter AB. A circle γ, whose center C lies on ω, is tangent to AB at D and cuts ω at E and F. Prove that triangles CEF and DEF have the same area.

Solution. Let segments CD and EF intersect at M. Extend CD to meet γ at G and ω at H, noting that $GC = CD = DH$. By Power of a Point,

$$MG \cdot MD = ME \cdot MF = MC \cdot MH$$
$$(CG + MC) \cdot MD = MC \cdot (MD + DH)$$
$$CG \cdot MD = MC \cdot DH$$
$$MD = MC.$$

So M is the midpoint of CD. Drop the perpendiculars CX and DY to EF. Triangles MCX and MDY are congruent by AAS, so the heights CX, DY are equal and the triangles CEF and DEF have the same area.

6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) + f(x+y) = f(x)f(y) + f(x) + f(y). \tag{3}$$

Solution. The answers are $f(x) = 0$ and $f(x) = x$. It is easy to check that both are valid.

Plugging $(0,0)$ into (3) immediately yields $f(0) = 0$. Plugging in $y = 1$ yields the relation

$$f(x+1) = f(1)[f(x) + 1]. \tag{4}$$

If $f(1) = 0$, then we conclude that f is identically 0. So we assume that $f(1) = t \neq 0$, so (4) uniquely determines $f(x+1)$ given $f(x)$ or vice versa. In this way we obtain the values

$$f(0) = 0, \quad f(1) = t, \quad f(2) = t^2 + t, \quad f(-1) = -1, \quad f(-2) = -1 - \frac{1}{t}.$$

Plugging the number pair $(2,-1)$ into (3) yields an equation in t that simplifies to $t^2 = 1$, i.e. $t = 1$ or $t = -1$.

The case $t = -1$ (which incidentally is tenable if f were defined only on the integers) can be disposed of in an ad hoc way. Plugging $(-1/2,-1)$ into (3) yields $f(-1/2) = -1/2$; two more applications of (4) give us $f(3/2) = -1/2$. It is now a simple matter to insert the number pair $(-1/2, 2)$ into (3) and obtain the contradiction $-3/2 = -1/2$, implying that the case $t = -1$ is impossible.

Therefore we have $t = 1$, and (4) simplifies to

$$f(x+1) = f(x) + 1. \tag{5}$$

By induction, $f(x) = x$ for all $x \in \mathbb{Z}$. Moreover, we claim that $f(x) = x$ for all $x \in \mathbb{Q}$. Let $x = p/q$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$. Plugging $(p/q, q)$ into (3) gives

$$\begin{aligned} f(p) + f\left(\frac{p}{q} + q\right) &= f\left(\frac{p}{q}\right) f(q) + f\left(\frac{p}{q}\right) + f(q) \\ p + f\left(\frac{p}{q} + q\right) &= qf\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + q. \end{aligned}$$

We can use (5) to replace $f(\frac{p}{q} + q)$ by $f(\frac{p}{q}) + q$, and then the equation is readily solved for $f(\frac{p}{q}) = \frac{p}{q}$.

Now we know that f is the identity on the rationals, and we would like to pass from the rationals to the reals using density. For this we need certain bounds. We first plug $(x,-1)$ into (3) and obtain

$$\begin{aligned} f(-x) + f(x-1) &= f(-1) \\ f(-x) + f(x) - 1 &= -1 \\ f(-x) &= -f(x), \end{aligned}$$

that is, f is an odd function. Now we plug in $(x,-x)$ and get

$$f(-x^2) = f(-x)f(x) + f(-x) + f(x) = -f(x)^2.$$

Thus f applied to a nonpositive number always yields a nonpositive result, and consequently f of a nonnegative number always yields a nonnegative result.

For our final step, we assume that y is a rational number in the interval $[-1,0]$ and x is any real number. Then (3) yields

$$\begin{aligned} f(x+y) - f(y) &= f(x)f(y) - f(x) - f(xy) \\ f(x+y) - y &= (y+1)f(x) - f(xy). \end{aligned}$$

Our choice of y ensures that both terms on the right-hand side have the same sign as x (except that they may be zero). Let us keep $x + y = r$, where $r \in [-1,0]$ is a fixed irrational number, and vary y. We observe that, for each $y \in \mathbb{Q} \cap [-1,0]$, the differences $f(r) - y$ and $r - y$ have the same sign. This can only happen if $f(r) = r$.

Thus f is the identity on $[-1,0]$, and by (5), it is the identity everywhere.

7. Let $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, let a_n be the smallest positive integer such that $a_n \neq a_i$ for $i < n$ and $\gcd(a_n, a_{n-1}) > 1$. Prove that every positive integer appears as some a_i.

Solution. Our solution will proceed in three steps.

Step 1. *We show that there is a prime p such that infinitely many of the terms a_i are divisible by p.*

Proof. Suppose that such a prime p does not exist. In particular, taking $p = 2$, we find that there is an N_1 such that for all $i \geq N_1$, a_i is odd. Now let p_1 be the largest prime dividing any of the numbers $a_1, \ldots, a_{N_1}$, and choose $N_2 > N_1$ such that for $i \geq N_2$, all prime factors of a_i exceed p_1. The terms a_{N_2} and a_{N_2+1} have a common prime divisor $p_2 > p_1$. Note that the term $2p_2$ never appears in the sequence, since all the even terms have prime factors bounded by p_1. Therefore p_2 never appears in the sequence, since if $a_n = p_2$ for some n, we would have $a_{n+1} = 2p_2$. Thus $a_{N_2+1} \geq 3p_2$, which is impossible, since $p_2 < a_{N_2+1}$ has not appeared at stage $N_2 + 1$ and shares a factor with a_{N_2}. □

Step 2. *We show that every multiple of p is in the sequence $\{a_i\}$.*

Proof. Suppose that $p, 2p, \ldots, (k-1)p$ are in the sequence but kp is not ($k \geq 1$). Choose N such that all positive integers less than kp, if they appear in the sequence at all, are among the terms $a_1, \ldots, a_N$. Then, using Step 1, choose $n \geq N$ such that $p|a_n$. What can a_{n+1} be? By hypothesis $a_{n+1} \geq kp$; we note that kp is not relatively prime to a_n and not equal to any of the terms $a_1, \ldots, a_n$. So $a_{n+1} = kp$, a contradiction. □

Step 3. *We show that every positive integer is in the sequence $\{a_i\}$.*

Proof. Suppose that $1, 2, \ldots, k-1$ are in the sequence but k is not ($k \geq 3$). Choose N such that $1, 2, \ldots, k-1$ appear among the terms $a_1, a_2, \ldots, a_N$. By Step 2, there is an $n \geq N$ such that a_n is divisible by kp, hence by k. Using an argument like that of the preceding step, we get $a_{n+1} = k$, a contradiction. □