Berkeley Math Circle
Monthly Contest 5
Due February 5, 2013

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Calculate, with proof, the last digit of
$$3^{3^{3^{3^{3}}}}.$$
Remark. Note that this means 3^(3^(3^(3^3))), not (((3^3)^3)^3)^3.

2. Prove that any triangle can be dissected into five isosceles triangles. (The pieces must not intersect except at their boundaries, and they must cover the given triangle.)

   Remark. Watch out for holes in your solution. Does anything go wrong when the triangle is right? Obtuse? Isosceles?

3. Alice and Bob play the following game on the whiteboard. First, Alice writes an odd number in binary on the board. Then, beginning with Bob, the players take turns modifying the number in one of two ways: subtracting 1 from it (preserving the binary notation), or erasing its last digit. When the whiteboard is blank, the last player to have played wins. Which player has a winning strategy?

   Remark. If you think Alice has a winning strategy, you should explain what her starting move should be and how she can win regardless of what Bob does. If you think Bob has a winning strategy, you should explain how he can win regardless of what Alice does.

4. Let $a \leq b \leq c \leq d$ be real numbers such that
$$a + b + c + d = 0 \quad \text{and} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 0.$$
Prove that $a + d = 0$.

5. Let ω be a circle with diameter AB. A circle γ, whose center C lies on ω, is tangent to AB at D and cuts ω at E and F. Prove that triangles CEF and DEF have the same area.

6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) + f(x+y) = f(x)f(y) + f(x) + f(y). \tag{1}$$

7. Let $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, let a_n be the smallest positive integer such that $a_n \neq a_i$ for $i < n$ and $\gcd(a_n, a_{n-1}) > 1$. Prove that every positive integer appears as some a_i.