Berkeley Math Circle
Monthly Contest 3 – Solutions

1. A number is written at each edge of a cube. The cube is called *magic* if:

   (i) For every face, the four edges around it have the same sum.

   (ii) For every vertex, the three edges meeting at it have the same sum. (The face and vertex sums may be different.)

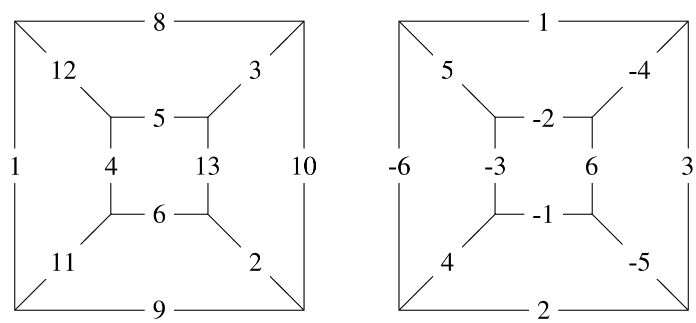
   Determine if there exists a magic cube using

   (a) the numbers 1 through 12, each only once;

   (b) the numbers from 1 through 13, each no more than once.

   Solution. (a) The answer is no. If the cube is magic, then every triple of edges abutting a vertex has of course the same average. Since every edge belongs to the same number of vertices (two), these vertex averages are the same as the average of all the numbers on the cube, which is $13/2$. But it is impossible for three numbers to have an average of $13/2$.

   (b) The answer is yes.



   On the right we have subtracted 7 from each number to clarify the structure of the solution. Every face and vertex adds to 0, as does each pair of opposite edges.

2. Define a *multiplication table* to be a rectangular array in which every row is labeled with a different positive integer, every column is labeled with a different positive integer, and every cell is labeled with the product of its row and column numbers, for instance:

$\times$	2	6	4	3
2	4	12	8	6
1	2	6	4	3
3	6	18	12	9

   The above table is 3×4 but contains only 8 distinct products. What is the minimum number of distinct products in a 2012×2012 multiplication table?

   Solution. The answer is $2\cdot2012-1=4013$.

   To get a multiplication table with only 4013 products, it suffices to fill the rows and columns with successive powers of two: $2^0, 2^1, 2^2, \ldots, 2^{2012}$ for both the rows and the columns. Then the products within the table will be $2^0, 2^1, \ldots, 2^{2\cdot2012}$.

   To prove that the number of products can be no fewer, let us rearrange the rows and columns of the table so that they are in increasing order. Then consider moving in an L-shaped path from the upper left product to the upper right product to the lower right product:

$\times$	2	3	4	6
1	:2:	:3:	:4:	:6:
2	4	6	8	:12:
3	6	9	12	:18:

   In a 2012×2012 table, this traverses 4013 cells, and all the while the products are increasing, so there are at least 4013 distinct products.

3. Triangle ABC is inscribed in a circle centered at O, and M is the midpoint of BC. Suppose that A, M, and O are collinear. Prove that $\triangle ABC$ is either right or isosceles (or both).

Solution. If M and O coincide, then BC is a diameter. Then $\angle A$ is right (this follows from the well-known theorem that an angle inscribed in a semicircle is a right angle).

If M and O do not coincide, then line MO must be the perpendicular bisector of BC since M and O are both equidistant from B and C. We are given that A lies on MO. So A is equidistant from B and C, i.e. $\triangle ABC$ is isosceles.

4. Let p be a prime number that has the form $a^3 - b^3$ for some positive integers a and b. Prove that p also has the form $c^2 + 3d^2$ for some positive integers c and d.

Solution. We can factor

$$p = a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

Since a and b are positive integers, the only way this can happen is if $a - b = 1$.

Either a or b is even. If a is even, let $a = 2u$, so $b = 2u - 1$. Then

$$\begin{aligned} p &= (2u)^2 + (2u)(2u-1) + (2u-1)^2 \\ &= 12u^2 - 6u + 1 \\ &= (3u-1)^2 + 3u^2 \end{aligned}$$

has the desired form. If b is even, let $b = 2u$, so $a = 2u + 1$. Then

$$\begin{aligned} p &= (2u+1)^2 + (2u)(2u+1) + (2u)^2 \\ &= 12u^2 + 6u + 1 \\ &= (3u+1)^2 + 3u^2 \end{aligned}$$

has the desired form.

5. (a) Prove that there are 2012 points on the unit circle such that the distance between any two of them is rational.

   (b) Does there exist an infinite set of points on the unit circle such that the distance between any two of them is rational?

Solution. The answer to part (b) is yes.

For brevity, we use the notation of complex numbers. For any integers x and y, not both zero, let $z = x + yi$ and

$$P(z) = \frac{z^2}{\bar{z}^2}.$$

Clearly, $|P(z)| = 1$ so $P(z)$ is a point on the unit circle. We claim that for any $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$, the distance between $P(z_1)$ and $P(z_2)$ is rational. We compute:

$$\begin{aligned} |P(z_1) - P(z_2)|^2 &= (P(z_1) - P(z_2))(\overline{P(z_1)} - \overline{P(z_2)}) \\ &= \left(\frac{z_1^2}{\bar{z}_1^2} - \frac{z_2^2}{\bar{z}_2^2}\right)\left(\frac{\bar{z}_1^2}{z_1^2} - \frac{\bar{z}_2^2}{z_2^2}\right) \\ &= -\frac{z_1^2\bar{z}_2^2}{\bar{z}_1^2 z_2^2} + 2 - \frac{\bar{z}_1^2 z_2^2}{z_1^2\bar{z}_2^2} \\ &= -\left(\frac{z_1\bar{z}_2}{\bar{z}_1 z_2} - \frac{\bar{z}_1 z_2}{z_1\bar{z}_2}\right)^2. \end{aligned}$$

The expression in parentheses is purely imaginary (being the difference of a complex number and its conjugate). Its imaginary part is rational since the components of z_1 and z_2 are rational. We conclude that $|P(z_1) - P(z_2)|$ is rational.

To finish the proof, it suffices to show that $P(z)$ takes on infinitely many values. It is not hard to check that the choices $z = 1 + i, 1 + 2i, 1 + 3i, \ldots$ all give distinct values of $P(z)$.

6. A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ of degree n or less is called *happy* if

   (i) Each coefficient a_i satisfies $0 \le a_i < 1$;

   (ii) $f(x)$ is an integer for all integers x.

Find the number of happy polynomials of degree n or less.

Solution. The answer is the "superfactorial"

$$1! \cdot 2! \cdots n! = 1^n 2^{n-1} \cdots n^1.$$

For $n = 1$, the result is clear. We will prove as an induction step that there are $n!$ times as many happy polynomials of degree at most n as of degree at most $n-1$.

Let $f(x)$ be a happy polynomial of degree at most n. We claim that the leading coefficient of f is $c/n!$ for some integer c. One way to see this is using "Lagrange interpolation": let

$$f_0(x) = \sum_{0 \le k \le n} \left(f(k) \prod_{\substack{0 \le j \le n \\ j \ne k}} \frac{x-j}{k-j} \right).$$

On the one hand, $f_0(x)$ is clearly a polynomial of degree at most n with rational coefficients. On the other hand, for each $x = 0, \ldots, n$, we have $f_0(x) = f(x)$ since all terms for $k \ne x$ are zero and the remaining term is $f(x) \cdot 1$. Therefore $f_0 = f$ since two distinct polynomials of degree n can have at most n common values. The leading coefficient of $f = f_0$ is a sum of terms of the form

$$\frac{1}{(-k)(-k+1)\cdots(-2)(-1)\cdot 1 \cdot 2 \cdots (n-k)} \cdot f(k) = \frac{\pm 1}{k!(n-k)!} f(k) = \frac{\pm\binom{n}{k}}{n!} f(k),$$

so it has the form $c/n!$ for $c \in \mathbb{Z}$.

By the given inequalities, we know that $0 \le c < n!$. Given $f(x)$, let

$$g(x) = f(x) - c \cdot \frac{x(x-1)\cdots(x-n+1)}{n!},$$

a polynomial of degree at most $n-1$. For integers $x \ge n$, the fraction in the definition of $g(x)$ is the integer $\binom{x}{n}$, from which it is easy to see that $g(x)$ is an integer for all integers x. Let $h(x)$ be the polynomial obtained by reducing each coefficient of g mod 1 so as to lie in the interval $[0, 1)$. Then $f(x) \mapsto h(x)$ defines a bijection between

- happy polynomials of degree $\le n$ with nth-degree coefficient $c/n!$, and
- happy polynomials of degree $\le n-1$.

Since there are $n!$ possible values of c, it follows that there are $n!$ times as many happy polynomials of degree $\le n$ as of degree $\le n-1$.

7. Show that for all real numbers a, b, c,

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

Solution. A solution will be coming soon.