Berkeley Math Circle
Monthly Contest 3
Due December 4, 2012

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 3
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. A number is written at each edge of a cube. The cube is called *magic* if:

   (i) For every face, the four edges around it have the same sum.

   (ii) For every vertex, the three edges meeting at it have the same sum. (The face and vertex sums may be different.)

   Determine if there exists a magic cube using

   (a) the numbers 1 through 12, each only once;

   (b) the numbers from 1 through 13, each no more than once.

2. Define a *multiplication table* to be a rectangular array in which every row is labeled with a different positive integer, every column is labeled with a different positive integer, and every cell is labeled with the product of its row and column numbers, for instance:

$\times$	2	6	4	3
2	4	12	8	6
1	2	6	4	3
3	6	18	12	9

   The above table is 3×4 but contains only 8 distinct products. What is the minimum number of distinct products in a 2012×2012 multiplication table?

3. Triangle ABC is inscribed in a circle centered at O, and M is the midpoint of BC. Suppose that A, M, and O are collinear. Prove that $\triangle ABC$ is either right or isosceles (or both).

4. Let p be a prime number that has the form $a^3 - b^3$ for some positive integers a and b. Prove that p also has the form $c^2 + 3d^2$ for some positive integers c and d.

5. (a) Prove that there are 2012 points on the unit circle such that the distance between any two of them is rational.

   (b) Does there exist an infinite set of points on the unit circle such that the distance between any two of them is rational?

6. A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ of degree n or less is called *happy* if

   (i) Each coefficient a_i satisfies $0 \le a_i < 1$;

   (ii) $f(x)$ is an integer for all integers x.

   Find the number of happy polynomials of degree n or less.

7. Show that for all real numbers a, b, c,

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$