Berkeley Math Circle
Monthly Contest 2
Due October 30, 2012

Instructions
This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 2
by Bart Simpson
Grade 5, BMC Beginner
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

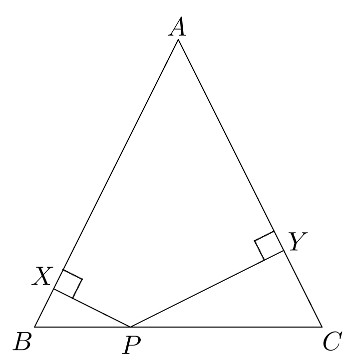
Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. There are 25 people at a party and every pair of them is either friends or strangers. Prove that there are two people at the party who have the same number of friends.

2. Let ABC be an isosceles triangle with $AB = AC$, and let P be a point moving along the side BC. Drop the heights PX, PY from P to the sides AB and AC. Prove that the sum $PX + PY$ remains constant as P moves.



3. Two fractions

$$\frac{a}{b} \quad \text{and} \quad \frac{c}{d}$$

are called *approximately equal* if a, b, c, d are positive integers and

$$\frac{a}{b} - \frac{c}{d} = \frac{1}{bd}.$$

Prove that given two approximately equal fractions, we can multiply the four numerators and denominators by the same positive integer and then add or subtract 1 from each of them so that the resulting fractions are equal. For instance, given the approximately equal fractions

$$\frac{1}{2} \quad \text{and} \quad \frac{2}{5},$$

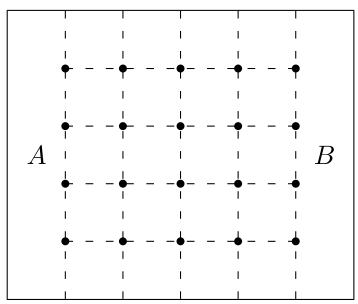
we can multiply each term by 10 to get

$$\frac{10}{20} \quad \text{and} \quad \frac{20}{50}$$

and then adjust each term by 1 to get the equality

$$\frac{9}{21} = \frac{21}{49}.$$

4. In the following maze, each of the dashed segments is randomly colored either black or white. What is the probability that there will exist a path from side A to side B that does not cross any of the black lines?



5. Prove that there are infinitely many primes p with the following property: there exists a positive integer k such that $2^k - 3$ is divisible by p.

6. Determine whether there exists a polynomial $f(x, y)$ of two variables, with real coefficients, with the following property: A positive integer m is a triangular number if and only if there do *not* exist positive integers x and y such that $f(x, y) = m$.

   Remark. A triangular number is one of the numbers $1, 3, 6, 10, \ldots$ of the form $\frac{n^2+n}{2}$, where n is a positive integer.

7. Let $ABCDE$ be a convex pentagon circumscribed around a circle ω such that $AB \parallel CD$ and $BC \parallel DE$. Locate points X and Y on rays AB and ED, respectively, such that $BX = AB$ and $DY = DE$. Prove that XY is tangent to ω.