Berkeley Math Circle
Monthly Contest 1 – Solutions

1. Find, with proof, all ways to write 1 as a sum of three fractions, each with numerator 1 and positive integer denominator. (The order of the fractions is irrelevant, so for instance $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}$ is the same as $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}$.)

   Solution. There are three solutions:

$$1=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}.$$

   Now we must prove that these are the only solutions. If the fraction $1/2$ appears in the expression, the remaining fractions must add to $1/2$, so one of them is greater than or equal to $1/4$. If this fraction is $1/3$, we get the solution $1/2+1/3+1/6$, and if this fraction is $1/4$, we get the solution $1/2+1/4+1/4$. Thus, if the fraction $1/2$ is used, we cannot get any new solutions.

   If the fraction $1/2$ does NOT appear in the expression, then all three fractions are at most $1/3$. Then their sum will certainly be less than 1 unless they are all equal to $1/3$. Thus in this case, we only get the third solution, $1/3+1/3+1/3$.

2. Determine whether there exists a number that begins with 2 having the property that, when the 2 is moved to the end, the number is

   (a) doubled;

   (b) tripled.

   Solution.

   There is no such number. Suppose that it existed; represent it by $2\cdots a\,b$ where a and b are the last two digits of the number. Let us write the digit-by-digit multiplication:

$$\begin{array}{cccccc} & 2 & \cdots & \cdots & a & b \\ & & & & \times & 2 \\ \hline \cdots & \cdots & a & b & 2 & \end{array}$$

   From the last column it is clear that there are only two choices for b: 1 and 6. We now look at the next-to-last column. If b is 1, there is no way that the multiplication of a by 2, without an added carry, can give the odd number 1. On the other hand, if b is 6, there is no way that the multiplication of b by 2, *with* the added carry of 1 resulting from $6\times 2=12$, can give the even number 6.

   The answer is yes; such a number is 285714, which satisfies $285714\times 3=857142$. There is no need to say more; we have rigorously proved that such a number exists. However, we will mention two of many methods for finding this number. The first is the end-digit analysis used in part (a):

$$\begin{aligned} \cdots x\times 3 &= \cdots 2 \text{ gives } x=4 \text{ (no other digit works)}\\ \cdots x4\times 3 &= \cdots 42 \text{ gives } x=1, \text{ etc.}\end{aligned}$$

   This process continues until we reach a number that begins with 2.

   The second method is an algebraic one. Suppose that the number is x and has n digits. Then moving a 2 from the beginning to the end of x can be achieved by first multiplying x by 10 (thus adding a 0 at the end), adding 2 (thus changing this 0 to a 2), and subtracting $2\cdot 10^n$ (deleting the 2 at the beginning). We have the condition

$$10x+2-2\cdot 10^n=3x,$$

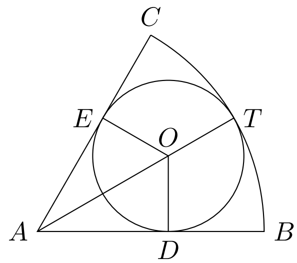
   which we can solve for x:

$$\begin{aligned} 10x-3x &= 2\cdot 10^n-2\\ 7x &= 2\cdot 10^n-2\\ x &= \frac{2\cdot 10^n-2}{7}.\end{aligned}$$

   Thus for x to exist, $2\cdot 10^n-2$ must be divisible by 7. As is easily verified, this does not hold for $n=1$, 2, 3, 4, or 5, but it works for $n=6$, yielding

$$x=\frac{2000000-2}{7}=285714.$$

3. A circle is inscribed in a sector that is one sixth of a circle of radius 6. (That is, the circle is tangent to both segments and the arc forming the sector.) Find, with proof, the radius of the small circle.



Solution. Because the circles are tangent, we can draw the line AT, which passes through the center O of the other circle. Note that triangles ADO and AEO are symmetric (this is HL congruence: AO is shared, radii OD and OE are equal, and angles ADO and AEO are right). Therefore, since $\angle BAC$ is $60°$, angles BAT and TAC are each $30°$. Now ADO is a $30°$-$60°$-$90°$ right triangle. If we let r be the radius of the small circle, then $OD = r$ and $OA = 2r$, but $OT = r$ so $AT = 3r$. But AT is the radius of the large circle, so $3r = 6$ and $r = 2$.

4. Call a positive integer *one-full* if it satisfies the following criteria:

   (a) Every digit is either 0, 1, or 2.

   (b) Out of every two consecutive digits, at least one of them is a 1.

   For $n \geq 2$, calculate the number of n-digit one-full numbers. (Numbers beginning with the digit 0 are not allowed.)

   Solution. The answer is 2^n. It is clear that there are four 2-digit one-full numbers: 10, 11, 12, and 21. To prove that the required number is 2^n for all $n \geq 3$, it is enough to show that there are twice as many n-digit one-full numbers as $(n-1)$-digit one-full numbers for every n. Given an $(n-1)$-digit one-full number, we construct two different n-digit one-full numbers via the following rules:

   - If the first digit is 1, we replace it with 11 and then with 21.
   - If the first digit is 2, we replace it with 12 and then with 10.

   Since 2's and 0's have the same function in the middle of a number, the two numbers that we get using this process are both one-full. Moreover, since any n-digit one-full number must begin with one of the four 2-digit one-full numbers, we can get all n-digit one-full numbers in this way. Thus there are twice as many n-digit one-full numbers as $(n-1)$-digit ones.

5. For integers $n \geq 1$, prove that the product
$$3 \cdot 12 \cdot 21 \cdot 30 \cdot \cdots \cdot (9n-6)$$
is divisible by $n!$.

   Solution. Let p be a prime number. We will prove that the number A of factors of p in $3 \cdot 12 \cdot \cdots \cdot (9n-6)$ is greater than or equal to the number B of factors of p in $n!$.

   We first explain the widely known method for computing B. Out of the numbers from 1 to n, exactly $\lfloor n/p \rfloor$ of them are multiples of p; they will contribute $\lfloor n/p \rfloor$ "first" factors of p to the product $n!$. In addition, $\lfloor n/p^2 \rfloor$ of these numbers are also divisible by p^2, giving $\lfloor n/p^2 \rfloor$ "second" factors of p. This continues, and we get
$$B = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots,$$
where the sum continues until eventually all of its terms become 0 due to a lack of terms divisible by very high powers of p.

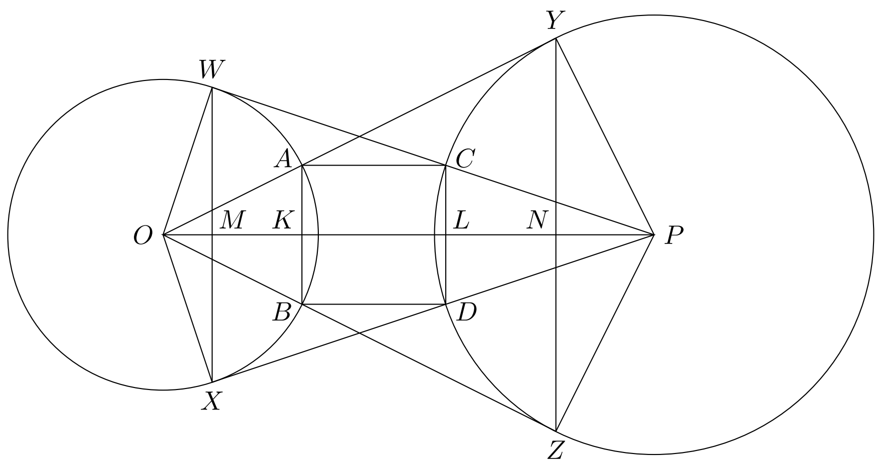
   Now we estimate A. If $p = 3$, then $A = n$ since every term of the arithmetic sequence $3, 12, 21, \ldots$ is divisible by 3 but not 9. In this case it is clear that
$$B = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{3^2} \right\rfloor + \left\lfloor \frac{n}{3^3} \right\rfloor + \cdots \leq \frac{n}{3} + \frac{n}{3^2} + \frac{n}{3^3} + \cdots = \frac{n}{2} < n.$$

   Now assume that $p \neq 3$. Divide the arithmetic progression $3, 12, 21, \ldots, (9n-6)$ into $\lfloor n/p \rfloor$ blocks of length p, discarding any terms that remain; because the common difference 9 is relatively prime to p, each block will have one representative of each congruence class mod p, and in particular exactly one multiple of p. Thus the product $3 \cdot 12 \cdot 21 \cdot 30 \cdot \dots \cdot (9n-6)$ will have at least $\lfloor n/p \rfloor$ "first" factors of p. By the same argument, using blocks of length p^2, there are at least $\lfloor n/p^2 \rfloor$ "second" factors of p, and so on, so

$$A \geq \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots = B,$$

as desired.

6. Circles j and k, centered at O and P respectively, do not intersect. The two tangent rays from O to k meet j at A and B, respectively, and the two tangent rays from P to j meet k at C and D, respectively. Prove that A, B, C, and D are the vertices of a rectangle.



Solution. Without loss of generality, we may assume that A, B, C, and D have the relative positions shown. We label the points of tangency W, X, Y, and Z. We also note that the entire construction is symmetric about about the line of centers OP, which therefore perpendicularly bisects segments AB, CD, WX, and YZ at their respective midpoints K, L, M, and N. Let r_1 and r_2 be the respective radii of j and k. We will first prove that $AK = CL$. Since $\triangle OAK \sim \triangle OYN \sim \triangle OPY$, we have

$$\frac{AK}{OA} = \frac{PY}{OP}, \quad \text{so } AK = \frac{OA \cdot PY}{OP} = \frac{r_1 r_2}{OP}.$$

Symmetrically, $CL = r_1 r_2 / OP$ so $AK = CL$. Now quadrilateral $AKLC$ has right angles at K and L and equal, parallel sides $AK = CL$, so it is a rectangle. Symmetrically, $BKLD$ is a rectangle so $ABDC$ is a rectangle.

7. In the hold of a pirate ship are ten treasure chests lying on pedestals in a circular arrangement. The captain would like to move each chest clockwise by one pedestal. However, the chests are so heavy that the captain and his assistant can only switch two chests at a time. What is the minimum number of switches needed to accomplish the task?

Solution. The answer is 9. It is easy to see that nine moves are sufficient; they can move a single chest counterclockwise one pedestal at a time, until after nine moves all of the other chests have been moved clockwise one pedestal.

Suppose that, at each stage of the game, we draw an arrow from each pedestal to the pedestal on which the chest which it now holds originally sat. Then every pedestal has one incoming and one outgoing arrow, and the overall graph can be decomposed into some number of disconnected cycles. Let c be the number of cycles. We note that at the beginning of the process, $c = 10$ (every arrow is its own cycle) while at the end, $c = 1$ (all the pedestals are connected in a ring).

We now claim that at each move, c cannot decrease by more than 1. Each move affects only two pedestals, which may initially belong to one or two cycles; the performance of the move may fuse two cycles into one or break up cycles into more cycles, but it certainly cannot replace two cycles by zero cycles. Therefore, c decreases by at most 1 at each move and hence cannot reach as low as 1 before the ninth move.

Remark. Math circlers with a penchant for enumerative combinatorics may wish to prove that there are precisely 100,000,000 ways of accomplishing the task in nine moves.