BAMO-8 Preparation
Berkeley Math Circle
February 21, 2012

This handout serves as a preparation for the BAMO-8. From 1999 to 2007 there was only one BAMO exam, and since 2008 there have been two: BAMO-8 and BAMO-12. The first three problems from previous BAMO exams roughly compare in difficulty level to the current BAMO-8.

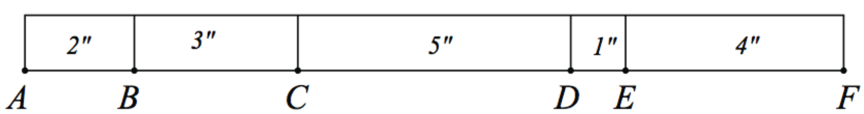
Be sure to also check out the BAMO archives where you can find all the previous handouts, and also the BAMO prep materials in the BMC archives. Good luck!

1. **(BAMO, 2003, Problem 1)** An integer is a *perfect* number if and only if it is equal to the sum of all of its divisors except itself. For example, 28 is a perfect number since $28 = 1 + 2 + 4 + 7 + 14$.

   Let $n!$ denote the product $1 \cdot 2 \cdot 3 \cdots n$, where n is a positive integer. An integer is a *factorial* if and only if it is equal to $n!$ for some positive integer n. For example, 24 is a factorial number since $24 = 4! = 1 \cdot 2 \cdot 3 \cdot 4$.

   Find all perfect numbers greater than 1 that are also factorials.

2. **(BAMO, 2007, Problem 1)** A 15-inch-long stick has four marks on it, dividing it into five segments of length $1, 2, 3, 4$, and 5 inches (although not neccessarily in that order) to make a "ruler." Here is an example. Using this ruler, you could measure 8 inches (between the marks B and D) and 11 inches (between the



   end of the ruler at A and the mark at E), but theres no way you could measure 12 inches. Prove that it is impossible to place the four marks on the stick such that the five segments have length $1, 2, 3, 4$, and 5 inches, and such that every integer distance from 1 inch through 15 inches could be measured.

3. **(BAMO, 2001, Problem 1)** Each vertex of a regular 17-gon is colored red, blue, or green in such a way that no two adjacent vertices have the same color. Call a triangle "multicolored" if its vertices are colored red, blue, and green, in some order. Prove that the 17-gon can be cut along nonintersecting diagonals to form at least two multicolored triangles. (A *diagonal* of a polygon is a a line segment connecting two nonadjacent vertices. Diagonals are called nonintersecting if each pair of them either intersect in a vertex or do not intersect at all.)

4. **(BAMO, 2008, Problem 1)** Call a year *ultra-even* if all of its digits are even. Thus $2000, 2002, 2004, 2006$, and 2008 are all ultra-even years. They are all 2 years apart, which is the shortest possible gap. 2009 is not an ultra-even year because of the 9, and 2010 is not an ultra-even year because of the 1.

   - In the years between the years 1 and 10000, what is the longest possible gap between two ultra-even years? Give an example of two ultra-even years that far apart with no ultra-even years between them. Justify your answer.
   - What is the second-shortest possible gap (that is, the shortest gap longer than 2 years) between two ultra-even years? Again, give an example, and justify your answer.

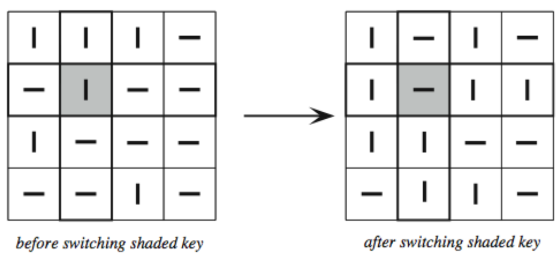
5. **(BAMO, 2003, Problem 2)** Five mathematicians find a bag of 100 gold coins in a room. They agree to split up the coins according to the following plan:

   - The oldest person in the room proposes a division of the coins among those present. (No coin may be split.) Then all present, including the proposer, vote on the proposal.

- If at least 50% of those present vote in favor of the proposal, the coins are distributed accordingly and everyone goes home. (In particular, a proposal wins on a tie vote.)
- If fewer than 50% of those present vote in favor of the proposal, the proposer must leave the room, receiving no coins. Then the process is repeated: the oldest person remaining proposes a division, and so on.
- There is no communication or discussion of any kind allowed other than what is needed for the proposer to state his or her proposal, and the voters to cast their vote.

Assume that each person wishes to maximize his or her share of the coins and behaves optimally. How much will each person get?

6. **(BAMO, 1999, Problem 3)** A lock has 16 keys arranged in a 4×4 array, each key oriented either horizontally or vertically. In order to open it, all the keys must be vertically oriented. When a key is switched to another position, all the other keys in the same row and column automatically switch their positions too (see figure). Show that no matter what the starting positions are, it is always possible to open this lock. (Only one key at a time can be switched.)



7. **(BAMO, 2000, Problem 3)** Let $x_1, x_2, \ldots, x_n$ be positive numbers, with $n \geq 2$. Prove that

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\cdots\left(x_n + \frac{1}{x_n}\right) \geq \left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{n-1} + \frac{1}{x_n}\right)\left(x_n + \frac{1}{x_1}\right).$$

8. **(BAMO, 2001, Problem 3)** Let $f(n)$ be a function satisfying the following three conditions for all positive integers n:

   (a) $f(n)$ is a positive integer,

   (b) $f(n+1) > f(n)$,

   (c) $f(f(n)) = 3n$.

   Find $f(2001)$.

9. **(BAMO, 2002, Problem 3)** A game is played with two players and an initial stack of n pennies ($n \geq 3$). The players take turns choosing one of the stacks of pennies on the table and splitting it into two stacks. The winner is the player who makes a move that causes all stacks to be of height 1 or 2. For which starting values of n does the player who goes first win, assuming best play by both players?

10. **(BAMO, 2004, Problem 3)** NASA has proposed populating Mars with 2004 settlements. The only way to get from one settlement to another will be by a connecting tunnel. A bored bureaucrat draws on a map of Mars, randomly placing N tunnels connecting the settlements in such a way that no two settlements have more than one tunnel connecting them. What is the smallest value of N that guarantees that, no matter how the tunnels are drawn, it will be possible to travel between any two settlements?

11. **(BAMO, 2007, Problem 3)** In $\triangle ABC$, D and E are two points on segment $\overline{BC}$ such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.