CONIC SECTIONS

VERA SERGANOVA

A straight line is the shortest path.

1. Ant Z is thirsty and tired. He wants to visit a straight stream nearby on his way home. Find the shortest path for him. (Given two points A, B and a line l not passing through A and B. Find a point X on l such that $AX + XB$ is the minimal possible.)

2. Ant Z is sitting at a vertex of a cubic box. He wants to get to the opposite vertex. Find the shortest part for him on the surface of the box.

3. Find the shortest path for Ant Z:

(a) from a point A to a point B both on a side of a cylindrical can;

(b) from a point A to a point B on an ice-cream cone;

(c) from a point A to a point B on a ball.

4. There are two balls, red and white, on the rectangular billiard table. You want to strike the red ball so that it hits two walls of the table first and then hits the white ball. How should you direct the red ball?

Ellipse. Fix two points A and B and the number $a > AB$. The set of all points X on the plane such that $AX + XB = a$ is called an *ellipse*. The points A and B are called the *foci* (plural of focus) of an ellipse. The distance between the foci of an ellipse is called the *focal distance*. In case when two foci coincide an ellipse becomes a circle. The *center* of an ellipse is the midpoint of the segment joining its foci. A *diameter* is a chord passing through the center.

All planets' orbits are ellipses with the sun at a focus.

5. An ellipse which is not a circle has two axes of symmetry: one is the line AB and another one is the perpendicular bisector to AB. The segments inside the ellipse cut by those lines are called the *major axis* and the *minor axis.*

6. Let a be the major axis, b be the minor axis and c be the focal distance. Show that $c^2 = a^2 - b^2$.

A *tangent line* to an ellipse is a line that intersects the ellipse at exactly one point. For any point X on an ellipse there exists a unique line through X tangent to the ellipse.

7. Let X be a point of an ellipse Γ, t be the tangent line to Γ at X. Let A and B be two foci of Γ. Show that the angle between AX and t equals the angle between BX and t.

Date: February 14, 2012.

Remark. The above problem illustrates a remarkable property of an elliptic mirror. If one places a light source at one focus of an ellipse, then all rays after reflecting at the elliptical mirror will pass through the second focus.

8. Let Δ be the circle with the major axis of the ellipse Γ as a diameter. Let A be a focus of Γ, X be a point on Δ and t be the line through X perpendicular to AX. Show that t is tangent to Γ.

9. Show that a section of an infinite cylinder by a plane Π not passing through the axis is an ellipse. For this inscribe two spheres in the cylinder so that they touch Π at the points A and B. (Imagine the cylinder as a vertical tube with the wall Π, drop one ball from above, push another from below.) Let X be a point of Π that lies on the cylinder. Let l be the line through X lying on the cylinder. Then l touches the spheres at some points P and Q. Show that $XP = XA$, $XQ = XB$, and therefore $XA+XB = PQ$. But the distance PQ does not depend on a choice of X. Therefore X lies on the ellipse with foci A and B and the major axis PQ.

10. Let X be a point on an ellipse Γ with center O, and t be the tangent line at X. Show that any chord parallel to t is bisected by the line OX.

Parabola. Let A and B be the foci of an ellipse Γ, and C be the point where Γ meets the line AB such that A is between C and B. Consider the family of all ellipses passing through C with one focus at A and another B' lying on the line CA. As the second focus B' moves further on the line CA so that AB' becomes larger, the family of ellipses approaches an unbounded curve, that is called a *parabola*. Let D be on the line AB, C the midpoint of DA and l be the perpendicular to AB through D.

Let us give a formal definition of a parabola. Given a line l and a point A not on l, the set of all points X equidistant from l and A is called a parabola. The point A is called the *focus* and the line l is called the *directrix* of the parabola.

11. Let X be a point on a parabola Γ and t be the tangent line to Γ at X. Let XP be the perpendicular from X to the directrix l and A be the focus of Γ. Show that the angle between t and XP equals the angle between t and AX.

Hyperbola. The definition of a hyperbola is very similar to the definition of an ellipse. Fix two points A and B and the number $a < AB$. The set of points X such that $AX - BX = a$ or $BX - AX = a$ is called a hyperbola with foci A and B. A hyperbola has two branches: the set of points X satisfying $AX - XB = a$ and the set of points Y satisfying $BY - AY = a$. The hyperbola is symmetric with respect to the line AB and the perpendicular bisector to AB.

12. Let X be a point on a hyperbola Γ with foci A and B and t be the tangent line to Γ at X. Show that t is the angle bisector of the angle AXB.

Let a hyperbola Γ have foci A and B, let O be the midpoint of AB. There are two lines through O that are called the *asymptotes* of the hyperbola Γ. They can be characterized by the following properties: Γ lies inside two vertical angles formed by the asymptotes, every ray with origin at O lying inside one of these angles meets Γ in one point.

Conic section. Take two lines l and s meeting at the point O. The surface obtained by rotating l about s is called a *cone*. The point O is called the *apex* of the cone. A line l' lying on a cone is called a *generatrix*. A cone has two halfs (above and below the apex). When we cut the cone by a plane perpendicular to s we obtain a circle (except the case when the plane passes through O). A *conic section* is a curve obtained by intersecting the cone with a plane not passing through O.

13. Show that any bounded conic section is an ellipse. To construct the foci consider two spheres inscribed into the cone and both tangent to the plane (see problem 9).

14. If a cutting plane Π is parallel to a generatrix of a cone, then the corresponding conic section is a parabola. There is a unique sphere inscribed into the cone and tangent to the plane. The point where the sphere touches Π is the focus of this parabola. To construct the directrix proceed as follows. The set of points where the cone touches the sphere is a circle. Let Π' be the plane containing this circle and l be the line of intersection of Π and Π'. Show that l is the directrix of the parabola.

15. If a cutting plane intersects both halfs of a cone, then the conic section is a hyperbola. Give a geometric construction of the foci by using inscribed spheres.

16. Following the construction of problem 14 show that for any conic section Γ, which is not a circle, and a fixed focus A of Γ there exists a line l and the number $k > 0$ such that for any point X on Γ the distance from X to l equals $k \cdot AX$. If $k < 1$, Γ is an ellipse, if $k > 1$, Γ is a hyperbola.

Projective transformation. Fix a point O in the space and two planes Π and Π' not passing through O. For a point X on Π let X' be the point where the line OX meets Π'.

17. Let m be the intersection of Π and the plane through O parallel to Π' and m' be the intersection of Π' and the plane through O parallel to Π. Show that X' is defined for all points of Π which do not belong to m, and any point Y of Π' not on m' equals X' for some point X of Π.

Consider the map $\Pi \to \Pi'$ which sends X to X'. When X gets "closer" and "closer" to a point X_0 on m, X' approaches the "infinite line" or the *horizon* of Π' in a certain direction. Different points of m correspond to different directions on Π'. If Y is on m we say that Y' is a point on the horizon Π'. Similarly, we say that the points on the horizon of Π go to the points of m' on Π'. The map we just constructed is called a projection $\Pi \to \Pi'$ with center O.

18. A projection maps a line to a line, and a conic section to a conic section.

Imagine that you are drawing a still life placed on a flat surface (a table). If you want to be realistic, a round plate will transfer to an ellipse at your picture. In fact, you are projecting the plane of the table to the plane of your picture. The center of this projection is the center of your pupil. (For best results keep one eye closed!).

Similarly, a ball usually have an elliptic shadow, and may become elliptic on a photo.

19. Given two conic sections Γ and Γ'. One can place them in the space so that Γ goes to Γ' under some projection. If Γ is a hyperbola and Γ' is an ellipse, where would the asymptotes of Γ go under this projection?

20. (Pascal's theorem) Let a hexagon $ABCDEF$ be inscribed into a conic section Γ. Let AB and DE meet at X, BC and EF meet at Y, CD and AF meet at Z. Then X, Y and Z lie on one line. One can use projections to reduce the problem to the case when Γ is a circle and the opposite sides of the hexagon are parallel.

21. (Brianchon's theorem). Let $ABCDEF$ be a hexagon formed by six tangent lines of a conic section. Then the diagonals AD, BE and CF meet at one point.

22. For any conic section formulate and prove the statement similar to Problem 10.

Equations. Let us use the coordinate system in the plane to define conic sections by equations.

23. Let an ellipse have foci at $(-\frac{c}{2}, 0)$ and $(\frac{c}{2}, 0)$, a and b be the major axis and the minor axis respectively. Show that it is define by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{4}$

24. Show that the area of an ellipse with axis a and b equals $\frac{\pi ab}{4}$, assuming the formula for the area of the circle.

25. Show that the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{4}$ defines a hyperbola. Find the foci and asymptotes of this hyperbola.

26. For any point X on a hyperbola consider the parallelogram formed by the asymptotes and the lines parallel to asymptotes passing through X. Prove that the area of this parallelogram does not depend on X.