Interrupted Games of Chance
Berkeley Math Circle (Advanced)
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1 The Problem

If a series of games (on which money has been bet) is interrupted before it can end, what is the fairest way to divide the stakes?

Rules of a simple version: 2 players A and B put 50$ each into a pot, and play a fair game repeatedly; the first one to win k times wins the whole pot, where k is some natural number.

Suppose the game is interrupted before completion. How should the pot be split up fairly?

History: In 1654, a French gambler named the Chevalier de Méré asked Blaise Pascal this question. Pascal and a friend of his wrote to each other, and each came up with a solution; different but both correct. This friend worked in other areas of math as well. Here's a quote of his about a different problem. . .

"Il est impossible de partager une puissance quelconque supérieure à 2 en deux puissances du même degré. . . J'en ai découvert une démonstration véritablement merveilleuse, que cette marge est trop étroite por contenir."

Exercise: Who might this be?

This led to what many consider to be the birth of probability theory.

1.1 Attempts at Solution

Some earlier attempts at solving this puzzle:

- **Solution 1:** Split the pot proportionally to the number of games won so far. If e.g. $k = 5$ and the score is $3 - 1$ for A, then he has won $3/4$ of the games so far, so he should get $3/4$ of the pot, i.e. $(3/4) \times 100\$ = 75\$$.

  Exercise: If $k = 20$ and the score is 19-10, how is the pot divided in this case? Does this seem fair?

- **Solution 2:** What should matter more is the number of *games left to win*, not how many they've won so far. A only needs 2 more wins, whereas B needs 4 more. So they should get an amount inversely proportional to how many games they have left to win. Since the number of tosses A needs to win is only half that of B, he should get twice as much money. This means (how?) that A gets $(2/3) \times 100\$ \approx 67\$$.

  Exercise: If $k = 20$ and the score is 19-10, how is the pot divided in this case? Does this seem fair?

Which solution is "better"? Is either any good? – Not well-defined questions.

1.2 The "Right Way"

Look at the **probability** that each player will go on to win the series, and divide stakes accordingly. I.e. if, when the game is interrupted, Player A has an 80% chance of eventually winning the series, then he should get 80% of the pot.

Is this necessarily the right answer? No. But it's a pretty reasonable way to do things. There's no perfect solution to this problem. These probabilities are what they show you on televised poker tournaments.

Exercise: In what mathematically precise way can we say that this is the right answer?

So how do we calculate these probabilities?

Define $p_k(x, y)$ to be the probability that player A is the first to k wins, given a current score of x to y.

2 Baseball

Let's focus on $k = 4$; for example, the World Series. Let's say A's (A) vs Giants (B). Assume every game is evenly matched and independent; this ignores: competitive imbalance, momentum, home-field advantage, pitching matchups,... Easy cases:

$$p_4(4, 0) = p_4(4, 1) = p_4(4, 2) = p_4(4, 3) = 1 \quad \text{(A's already won)}$$

and

$$p_4(0, 4) = p_4(1, 4) = p_4(2, 4) = p_4(3, 4) = 0 \quad \text{(Giants already won).}$$

These are the *boundary conditions*.

Also easy are $p_4(0, 0) = p_4(1, 1) = p_4(2, 2) = p_4(3, 3) = 1/2$; the series has to end, and everything is symmetric.

What about non-obvious ones, like $p_4(2, 1)$? General recurrence, for any k:

$$p_k(x, y) = (1/2)p_k(x + 1, y) + (1/2)p_k(x, y + 1), \quad p_k(k, y) = 1, \quad p_k(x, k) = 0 \quad \text{for all } x, y < k.$$

Geometric view: evolution of the score is a (random) walk on the lattice, up and/or to the right, until reaching boundary points. Go to point (x, y) on the lattice if the current score is x to y. Draw the lattice for the World Series!

Exercise: How important is the first game? I.e., find $p_4(1, 0)$. Is the first game more important than any other with this model?

2.1 Competitive Imbalance

What if teams are not evenly matched, and A's win each game with some probability $h \in (0, 1)$? No big deal. New recursion is

$$p_k(x, y) = hp_k(x + 1, y) + (1 - h)p_k(x, y + 1); \quad p_k(k, y) = 1, \quad p_k(x, k) = 0 \quad \text{for all } x, y < k.$$

and you can fill in the lattice the same way.

2.2 Explicit solution?

This is how Pascal solved the problem in general. Unfortunately, the ultimate solution is not as tidy as one would like.

$e(x, y) :=$ number of future outcomes that result in A winning, given a current score of x to y.

# of games left: $(2k-1) - x - y \;\Rightarrow$ number of possible future outcomes: $2^{(2k-1)-x-y}$

Wait. Where did $2k-1$ come from? How can we know how many games are left?

Recurrence for e:

$$e(x, y) = e(x+1, y) + e(x, y+1)$$

Boundary conditions:

$$e(x, 2k-1-x) = \begin{cases} 1 & \text{if } x \geq k \\ 0 & \text{if } x < k \end{cases}$$

(This is somewhat similar to the recursion for **Pascal's Triangle**:

$$\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$$

or, writing $b(n,k) = \binom{n}{k}$,

$$b(n,k) = b(n, k-1) + b(n-1, k-1), \quad b(n, 0) = 1 \quad \text{for all } n.)$$

The probability of A winning is $p_k(x, y) = e(x, y)/2^{(2k-1)-x-y}$ **if each game is fair** (i.e. $h = 1/2$), meaning any sequence of wins and losses is equally likely.

$$\begin{aligned} p_k(x,y) &= \frac{e(x,y)}{2^{2k-1-x-y}} \\ &= \frac{\#\text{ ways for A to win at least } k-x \text{ games out of } (2k-1)-x-y \text{ games}}{\#\text{ ways to play } 2k-1-x-y \text{ games}} \\ &= \sum_{j=k-x}^{2k-1-x-y} \binom{2k-1-x-y}{j} \Big/ 2^{2k-1-x-y} \\ &= \sum_{j=k-x}^{2k-1-x-y} \binom{2k-1-x-y}{j} \left(\frac{1}{2}\right)^{2k-1-x-y} \end{aligned}$$

There's however no tidy formula for $p_k(x, y)$ or $e_k(x, y)$ like there is for Pascal's triangle...

Exercise: In an evenly matched game to $k = 20$, Find the probability that A wins, given a current score of 19-10. **Hint**: this is not as hard as a general case.

Exercise: (Hard): Find some formula for $p_k(x, y)$, in the case where $h \neq 1/2$.

3 Tennis

What about a game of tennis? Rules for a game: first to score 4 points wins the game, but you **must win by two**. Assume skills are uneven: Player A has probability h of winning any point. As before, $p(x, y) =$ Probability of A winning given a current score of x to y. Ah. What's the new wrinkle here?

Recurrence is similar.

$$p(x,y) = hp(x+1,y) + (1-h)p(x,y+1); \quad p(4,y) = 1, \quad p(x,4) = 0 \qquad \text{for all } x,y \leq 2.$$

But we don't have enough boundary conditions as stated. What about $p(3,2)$, for example?

Trick:

$$p(3,3) = hp(4,3) + (1-h)p(3,4)$$
$$p(3,3) = h\Big(p(5,3) + (1-h)p(4,4)\Big) + (1-h)\Big(p(4,4) + (1-h)p(5,3)\Big)$$
$$p(3,3) = h(1 + (1-h)p(4,4)) + (1-h)(hp(4,4) + 0)$$

Seem to be too many unknowns. But notice: common sense dictates that $p(3,3)$ and $p(4,4)$ must be equal!

So

$$p(3,3) = h \cdot (h + (1-h)p(3,3)) + (1-h) \cdot (hp(3,3))$$

or

$$p(3,3) = \frac{h^2}{1 - 2h(1-h)}$$

This closes the grid and allows us to find every probability recursively. This was the solution of James Bernoulli in his 1713 treatise *Ars Conjectandi.*

Exercise: Find $p_4(0,0)$ for $h = 1/3$ and $h = 1/2$.

Exercise to think about: Do we have to worry about the possibility that the game **never** ends?

Exercise: (way too hard) find some (nontrivial) scheme that allows a game of tennis to go on forever with nonzero probability.

3.1 Extension 1

What if we don't know the relative skill from the outset, but we have no reason to think that the skills are equal, and we're going to figure out the relative skills as we go along? Here's an idea: for a current score, assume that the relative strengths of the players is proportional to the number of points they've won. This is a "Rich get richer" type of scheme. E.g. if the score is 3-1, our best guess is that Player A is better, and wins each point with probability 3/4. What's wrong with this? Anything? Um, yes. How do you get started? Do you update the probabilities at every step?

Trying to guess probabilities based on observed results: the definition of **Statistics**.

Exercise: Again suppose the score is interrupted at 19-10 in a game to $k = 20$. We determine that the probability of A winning any future point is equal to the proportion of points he's won so far. With this assumption, how likely is A to win, and how much of the pot does he get? Is this more or less likely than previous cases?

3.2 Extension 2

What if there is "momentum", in that the player who won the last point is more likely to win the current one? You can still make the grid, but you have to keep track of the path you're on, and not just the current score. E.g. $p_k(x, y)$ does not have enough information.

Exercise: Suppose in a first-to-win-3-games series, the first game is evenly matched, and the team that won the previous game has probability 0.6 of winning the current game. If the score is 1-1, find the probability of A winning if (a) A won the last game and (b) if B won the last game.

Exercise: You're in trouble: you owe n dollars to the mob, but you only have x dollars, and unfortunately, $x < n$. In a last-ditch effort to come up with enough cash, you go to a casino and repeatedly bet \$1 on an even-money game that wins with probability h. You play until you get the \$$n$ needed, or you go broke (and presumably go into hiding). You want to know the probability of getting that \$$n$, starting with \$$x$; call the probability $p(x)$.

1. Find a recursion like before to express $p(x)$, and make sure you include boundary conditions.
2. When $h = 1/2$, can you guess an explicit solution for $p(x)$?
3. **Harder** If $h \neq 1/2$, can you find an explicit solution?
4. Suppose $h = 1/3$, and $x = 10$, $n = 40$. Which is the better strategy: bet \$1 at a time, or bet everything you have at any time? What if $h = 2/3$? What if $h = 1/2$?