

AMC PREPARATION
BERKELEY MATH CIRCLE
JANUARY 31, 2012

This handout serves as a preparation for the AMC12. The best way to prepare is to practice, practice, practice, and hence this handout contains many problems from previous AMC and AHSME (the predecessor of AMC) exams. I've loosely grouped the problems by category. I've mainly included problems from the second half of the exam, which are harder, so don't worry if you can't solve all of them. I also highly recommend doing full exams from previous years while timing yourself; this tests your ability to work under time-pressure, which is a major issue when taking the AMC, since you only have 3 minutes per problem on average. You can find most of the previous exams online or in books at the library (e.g. the UC Berkeley Mathematics and Statistics Library). I've mainly included problems from old exams, so that you can practice the newer exams as a full exam.

COMBINATORICS

1. (AMC12A, 2003, Problem 20) How many 15-letter arrangements of 5 A 's, 5 B 's, and 5 C 's have no A 's in the first 5 letters, no B 's in the next 5 letters, and no C 's in the last 5 letters?

(A) $\sum_{k=0}^5 \binom{5}{k}^3$ (B) $3^5 \cdot 2^5$ (C) 2^{15} (D) $\frac{15!}{(5!)^3}$ (E) 3^{15}

2. (AHSME, 1989, Problem 22) A child has a set of 96 distinct blocks. Each block is one of 2 materials (*plastic, wood*), 3 sizes (*small, medium, large*), 4 colors (*blue, green, red, yellow*), and 4 shapes (*circle, hexagon, square, triangle*). How many blocks in the set are different from the "*plastic medium red circle*" in exactly two ways? (The "*wood medium red square*" is such a block.)

(A) 29 (B) 39 (C) 48 (D) 56 (E) 62

3. (AHSME, 1989, Problem 25) In a certain cross-country meet between two teams of five runners each, a runner who finishes in the n^{th} position contributes n to his team's score. The team with the lower score wins. If there are no ties among the runners, how many different **winning** scores are possible?

(A) 10 (B) 13 (C) 27 (D) 120 (E) 12626

GEOMETRY

1. (AHSME, 1983, Problem 19) Point D is on side CB of triangle ABC . If

$$\angle CAD = \angle DAB = 60^\circ, \quad AC = 3 \quad \text{and} \quad AB = 6,$$

then the length of AD is

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E) 4

2. (AHSME, 1983, Problem 27) A large sphere is on a horizontal field on a sunny day. At a certain time the shadow of the sphere reaches out a distance of $10m$ from the point where the sphere touches the ground. At the same instant a meter stick (held vertically with one end on the ground) casts a shadow of length $2m$.

What is the radius of the sphere in meters? (Assume the sun's rays are parallel and the meter stick is a line segment.)

- (A) $\frac{5}{2}$ (B) $9 - 4\sqrt{5}$ (C) $8\sqrt{10} - 23$ (D) $6 - \sqrt{15}$ (E) $10\sqrt{5} - 20$

3. (AHSME, 1983, Problem 29) A point P lies in the same plane as a given square of side 1. Let the vertices of the square, taken counterclockwise, be A, B, C and D . Also, let the distances from P to A, B and C , respectively, be u, v and w . What is the greatest distance that P can be from D if $u^2 + v^2 = w^2$?

- (A) $1 + \sqrt{2}$ (B) $2\sqrt{2}$ (C) $2 + \sqrt{2}$ (D) $3\sqrt{2}$ (E) $3 + \sqrt{2}$

4. (AHSME 1984, Problem 18) A point (x, y) is to be chosen in the coordinate plane so that it is equally distant from the x -axis, the y -axis, and the line $x + y = 2$. Then x is

- (A) $\sqrt{2} - 1$ (B) $\frac{1}{2}$ (C) $2 - \sqrt{2}$ (D) 1 (E) not uniquely determined

5. (AHSME, 1986, Problem 19) A park is in the shape of a regular hexagon 2km on a side. Starting at a corner, Alice walks along the perimeter of the park for a distance of 5km. How many kilometers is she from her starting point?

- (A) $\sqrt{13}$ (B) $\sqrt{14}$ (C) $\sqrt{15}$ (D) $\sqrt{16}$ (E) $\sqrt{17}$

6. (AHSME, 1986, Problem 29) Two of the altitudes of the scalene triangle ABC have length 4 and 12. If the length of the third altitude is also an integer, what is the biggest it can be?

- (A) 4 (B) 5 (C) 6 (D) 7 (E) none of these

7. (AHSME, 1987, Problem 22) A ball was floating in a lake when the lake froze. The ball was removed (without breaking the ice), leaving a hole 24cm across at the top and 8cm deep. What was the radius of the ball (in centimeters)?

- (A) 8 (B) 12 (C) 13 (D) $8\sqrt{3}$ (E) $6\sqrt{6}$

8. (AHSME, 1987, Problem 27) A cube of cheese $C = \{(x, y, z) \mid 0 \leq x, y, z \leq 1\}$ is cut along the planes $x = y, y = z$ and $z = x$. How many pieces are there? (No cheese is moved until all three cuts are made.)

- (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

PROBABILITY

1. (AMC12A, 2003, Problem 16) A point P is chosen at random in the interior of the equilateral triangle ABC . What is the probability that $\triangle ABP$ has a greater area than each of $\triangle ACP$ and $\triangle BCP$?

- (A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) $\frac{2}{3}$

2. (AMC12B, 2003, Problem 25) Three points are chosen randomly and independently on the perimeter of a circle. What is the probability that all three pairwise distances between the points are less than the radius of the circle?

- (A) $\frac{1}{36}$ (B) $\frac{1}{24}$ (C) $\frac{1}{18}$ (D) $\frac{1}{12}$ (E) $\frac{1}{9}$

3. (AHSME, 1984, Problem 19) A box contains 11 balls, numbered $1, 2, 3, \dots, 11$. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers on the balls drawn is odd?

(A) $\frac{100}{231}$ (B) $\frac{115}{231}$ (C) $\frac{1}{2}$ (D) $\frac{118}{231}$ (E) $\frac{6}{11}$

4. (AHSME, 1985, Problem 24) A non-zero digit is chosen in such a way that the probability of choosing digit d is $\log_{10}(d+1) - \log_{10}d$. The probability that the digit 2 is chosen is exactly $1/2$ the probability that the digit chosen is in the set

(A) $\{2, 3\}$ (B) $\{3, 4\}$ (C) $\{4, 5, 6, 7, 8\}$ (D) $\{5, 6, 7, 8, 9\}$ (E) $\{4, 5, 6, 7, 8, 9\}$

5. (AHSME, 1987, Problem 26) The amount 2.5 is split into two nonnegative real numbers uniformly at random, for instance, into 2.143 and .357, or into $\sqrt{3}$ and $2.5 - \sqrt{3}$. Then each number is rounded to its nearest integer, for instance, 2 and 0 in the first case above, 2 and 1 in the second. What is the probability that the two integers sum to 3?

(A) $1/4$ (B) $2/5$ (C) $1/2$ (D) $3/5$ (E) $3/4$

6. (AHSME, 1988, Problem 28) An unfair coin has probability p of coming up heads on a single toss. Let w be the probability that, in 5 independent tosses of this coin, heads come up exactly 3 times. If $w = 144/625$, then

(A) p must be $2/5$ (B) p must be $3/5$ (C) p must be greater than $3/5$
(D) p is not uniquely determined (E) there is no value of p for which $w = 144/625$

7. (AMC12A, 2005, Problem 23) Two distinct numbers a and b are chosen randomly from the set $\{2, 2^2, 2^3, \dots, 2^{25}\}$. What is the probability that $\log_a b$ is an integer?

(A) $\frac{2}{25}$ (B) $\frac{31}{300}$ (C) $\frac{13}{100}$ (D) $\frac{7}{50}$ (E) $\frac{1}{2}$

8. (AMC12B, 2005, Problem 25) Six ants simultaneously stand on the six vertices of a regular octahedron, with each ant at a different vertex. Simultaneously and independently, each ant moves from its vertex to one of the four adjacent vertices, each with equal probability. What is the probability that no two ants arrive at the same vertex?

(A) $\frac{5}{256}$ (B) $\frac{21}{1024}$ (C) $\frac{11}{512}$ (D) $\frac{23}{1024}$ (E) $\frac{3}{128}$

OTHER

1. (AHSME, 1983, Problem 11) Simplify $\sin(x-y)\cos y + \cos(x-y)\sin y$.

(A) 1 (B) $\sin x$ (C) $\cos x$ (D) $\sin x \cos 2y$ (E) $\cos x \cos 2y$

Comment: try to solve this problem without any trigonometric identities. Hint: choose specific values of x and/or y .

2. (AMC12A, 2003, Problem 24) If $a \geq b > 1$, what is the largest possible value of $\log_a(a/b) + \log_b(b/a)$?

(A) -2 (B) 0 (C) 2 (D) 3 (E) 4

3. (AHSME, 1984, Problem 29) Find the largest value of $\frac{y}{x}$ for pairs of real numbers (x, y) which satisfy

$$(x - 3)^2 + (y - 3)^2 = 6.$$

- (A) $3 + 2\sqrt{2}$ (B) $2 + \sqrt{3}$ (C) $3\sqrt{3}$ (D) 6 (E) $6 + 2\sqrt{3}$

4. (AHSME, 1984, Problem 30) For any complex number $w = a + bi$, $|w|$ is defined to be the real number $\sqrt{a^2 + b^2}$. If $w = \cos 40^\circ + i \sin 40^\circ$, then

$$|w + 2w^2 + 3w^3 + \dots + 9w^9|^{-1}$$

equals

- (A) $\frac{1}{9} \sin 40^\circ$ (B) $\frac{2}{9} \sin 20^\circ$ (C) $\frac{1}{9} \cos 40^\circ$ (D) $\frac{1}{18} \cos 20^\circ$ (E) none of these

5. (AHSME, 1985, Problem 21) How many integers x satisfy the equation

$$(x^2 - x - 1)^{x+2} = 1?$$

- (A) 2 (B) 3 (C) 4 (D) 5 (E) none of these

6. (AHSME, 1985, Problem 25) The volume of a certain rectangular solid is 8cm^3 , its total surface area is 32cm^2 , and its three dimensions are in geometric progression. The sum of the lengths in cm of all edges of this solid is

- (A) 28 (B) 32 (C) 36 (D) 40 (E) 44

7. (AHSME, 1985, Problem 27) Consider a sequence x_1, x_2, x_3, \dots , defined by:

$$x_1 = \sqrt[3]{3},$$
$$x_2 = \left(\sqrt[3]{3}\right)^{\sqrt[3]{3}},$$

and in general

$$x_n = (x_{n-1})^{\sqrt[3]{3}} \quad \text{for } n > 1.$$

What is the smallest value of n for which x_n is an integer?

- (A) 2 (B) 3 (C) 4 (D) 9 (E) 27

8. (AHSME, 1986, Problem 23) Let $N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1$. How many positive integers are factors of N ?

- (A) 3 (B) 5 (C) 69 (D) 125 (E) 216

9. (AHSME, 1986, Problem 25) If $\lfloor x \rfloor$ is the greatest integer less than or equal to x , then

$$\sum_{N=1}^{1024} \lfloor \log_2 N \rfloor =$$

- (A) 8192 (B) 8204 (C) 9218 (D) $\lfloor \log_2(1024!) \rfloor$ (E) none of these

10. (AHSME, 1987, Problem 23) If p is a prime and both roots of $x^2 + px - 444p = 0$ are integers, then

- (A) $1 < p \leq 11$ (B) $11 < p \leq 21$ (C) $21 < p \leq 31$ (D) $31 < p \leq 41$ (E) $41 < p \leq 51$

11. (AHSME, 1987, Problem 24) How many polynomial functions f of degree ≥ 1 satisfy

$$f(x^2) = (f(x))^2 = f(f(x))?$$

- (A) 0 (B) 1 (C) 2 (D) finitely many but more than 2 (E) infinitely many

12. (AHSME, 1988, Problem 30) Let $f(x) = 4x - x^2$. Given x_0 , consider the sequence defined by $x_n = f(x_{n-1})$ for all $n \geq 1$. For how many real numbers x_0 will the sequence x_0, x_1, x_2, \dots take on only a finite number of different values?

- (A) 0 (B) 1 or 2 (C) 3, 4, 5 or 6 (D) more than 6 but finitely many (E) infinitely many