BAMO Preparation

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1. (China 1996) Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

2. (BAMO 2000) Finitely many cards are places in two stacks, with more cards in the left stack than the right. Each card has one or more distinct names written on it, although different cards may share some names. For each name, we define a "shuffle" by moving every card that has this name written on it to the opposite stack. Prove that it is always possible to end up with more cards in the right stack by picking several distinct names, and doing in turn the shuffle corresponding to each name.

3. (BAMO 2001) Let $f(n)$ be a function satisfying the following three conditions for all positive integers n:

   (a) $f(n)$ is a positive integer,

   (b) $f(n + 1) > f(n)$,

   (c) $f(f(n)) = 3n$.

   Find $f(2001)$.

4. (BAMO 2002) Professor Moriarty has designed a "prime-testing trail". The trail has 2002 stations, labeled 1...2002. Each station is colored either red or green, and contains a table which indicates, for each of the digits 0...9, another station number. A student is given a positive integer n, and then walks along the trail, starting at station 1. The student reads the first (leftmost) digit of n, and looks this digit up in station 1's table to get a new station location. The student then walks to this new station, reads the second digit of n and looks it up in that station's table to get yet another station location, and so on, until the last (rightmost) digit of n has been read and looked up, sending the student to his or her final station.

   Professor Moriarty claims that for any positive integer n, the final station will be green if and only if n is prime. Is this possible?

5. (BAMO 2004) A *tiling* of the plane with polygons consists of placing the polygons in the plane so that interiors of polygons do not overlap, each vertex of one polygon coincides with a vertex of another polygon, and no point of the plane is left uncovered. A *unit* polygon is a polygon with all sides of length one.

   (a) Prove that there is a tiling of the plane with infinitely many unit squares and infinitely many unit equilateral triangles in the same tiling.

   (b) Prove that it is impossible to find a tiling of the plane with infinitely many unit squares and finitely many (and at least one) unit equilateral triangles in the same tiling.

6. (BAMO 2004) Suppose one is given n real numbers, not all zero, but such that their sum is zero. Prove that one can label these numbers $a_1, a_2, \ldots a_n$ in a manner such that

$$a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n + a_na_1 < 0.$$

7. (BAMO 2004) Find (with proof) all monic polynomials $f(x)$ with integer coefficients that satisfy the following two conditions.

   (a) $f(0) = 2004$

   (b) If x is irrational, then $f(x)$ is also irrational.

8. (BAMO 2005) Let D be a dodecahedron inscribed in a sphere with radius R. Let I be an icosahedron which can also be inscribed in a sphere of radius R. Which has the greater volume, and why?

9. (BAMO 2007) Let N be the number of ordered pairs (x, y) of integers such that

$$x^2 + xy + y^2 \leq 2007.$$

   Remember, integers may be positive, negative, or zero!

   (a) Prove that N is odd.

   (b) Prove that N is not divisible by 3.

10. (BAMO 2008) A positive integer N is called *stable* if it is possible to split the set of all positive divisors of N (including N and 1) into two subsets that have no elements in common, but which have the same sum. For example, 6 is stable, because $1 + 2 + 3 = 6$, but 10 is not stable. Is $2^{2008} \cdot 2008$ stable?

11. (BAMO 2011) Does there exist a row of Pascal's triangle containing four distinct values a, b, c, and d such that $b = 2a$ and $d = 2c$?