Rigid motions of space. Dihedral Groups, Symmetric Groups, and Matrix Groups
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Let D_{2n} be Dihedral group, which is the group of symmetries of regular n-gon.

Let r be a rotation by $2\pi/n$ radians, and s be a reflection about line of symmetry through 1 and origin. Order of D_{2n} is $2n$.

1) Show $D_{2n} = \{1, r, r^2, ..., r^{n-1}, s, sr, sr^2, ..., sr^{n-1}\}$ and $D_{2n} =< r, s | r^n = s^2 = 1, rs = sr^{-1} >$ is a presentation of this group.

2) Find the number of symmetries of a parallelogram, a square, an equilateral triangle, a regular tetrahedron, a cube, and a dodecahedron.

3). Show that the group of symmetries of a cube contains a group of symmetries of a tetrahedron. (inscribe a regular tetrahedron in a cube.)

4). List the angles of rotations for all rotations which are symmetries of a cube, a dodecahedron.

Let S_n be the group of all permutations on n letters. Cycle decomposition of each permutation is a unique way of writing a permutation as a product of disjoint cycles.

1) Order of S_n is $n!$.

Theorem 0.1 *The order of a permutation is l.c.m. of the lengths of cycles its cycle decomposition.*

2) Let σ be a 12-cycle. For which i, σ^i is also 12-cycle. Show that if σ is a m-cycle, than σ^i is an m-cycle if and only if i is relatively prime to m.

3) Write down the cycle decomposition of each element of order 2 in S_4.

4) Let p be a prime number. Show that an element in S_n has order p if and only if its cycle decomposition is a product of disjoint p-cycles. Show its not true, if p is not prime.

5) Find the number of 3-cycles in S_4, the number of 4-cycles in S_7.

6) Find the number of permutations in S_5 that are product of two disjoint 2-cycles.

7) Every element can be written as a product of 2-cycles (transpositions). The number of this transpositions is odd even, we will call the sign of permutation +1 or -1 correspondingly.

8)Fact: m-cycle is an odd permutation if and only if m is even. Show that for any $\sigma \in S_n$, σ^2 is even.

9) Construct a homomorphism of S_4 onto S_3. (Think of S_4 as the symmetry group of the tetrahedron.)

10) Show that the rotation group of the cube is S_4. Show that the rotation group of the dodecahedron with the group of all even permutations in S_5.

Matrices as Rigid motions of $\mathbb{R}^2$. A rigid motion is a map of a plane to itself which preserves distances and angles. 2×2 matrices, their multiplication, inverses, identity, determinant, transpose.

1) Show multiplication is not commutative.
2) Show that det(AB)=det(A)det(B).
3) Find a matrix that represents a rotation about origin of $\mathbb{R}^2$ by t degrees, reflection about line with slope t. Other motions in $\mathbb{R}^2$ include translation, projection on x-coordinate, y-coordinate, reflection about y-axis, about x-axis. Can we find matrices corresponding to them? Find matrices that are rigid motions.
4) Orthogonal matrix is a matrix such that $AA^t = I = A^tA$. Prove that any orthogonal 2×2 matrix is a rotation or reflection. Do orthogonal matrices commute?
5) Find the determinant of an orthogonal matrix.