Inequalities in Elementary Geometry

Math Contest Preparation, Advanced Level

Berkeley Math Circle, 10/25/2011

Notation: In triangle $\triangle ABC$, $a = BC$, $b = CA$, $c = AB$.

1. Find $\triangle ABC$ with given base AB and fixed perimeter $p > 2 \cdot AB$ such that the area is as large as possible.

2. Find the largest possible square that can be covered by two discs of radius 1.

3. (Fagnano's Problem) In an acute triangle $\triangle ABC$, find the inscribed triangle that minimizes its perimeter.

4. (Fermat's problem) Given three points A, B, C in the plane, find the point P that minimizes $PA + PB + PC$.

5. (Erdös-Mordell inequality) Suppose A', B', C' are respectively in the sides BC, CA, AB of a triangle and P is in the interior of the triangle, show that $PA + PB + PC \geq 2(PA' + PB' + PC')$.

6. For an arbitrary point P in the interior of a triangle $\triangle ABC$, show that $PA + PB + PC \leq \max(a + b, b + c, c + a)$.

7. Suppose two points E in AB and F in AC satisfy that EF passes through the barycenter G of $\triangle ABC$, show that $EG \leq 2GF$.

8. Four pines stand at the corners of a square, Mr. Squirrel proposes to build a road system consisting of the two diagonals.

   (i) Can you defeat Mr. Squirrel by finding a better system (i.e. the total length is shorter) that connects all the pines?

   (ii) Compare your system with your friends'. Is your system optimal?

9. (Weitzenböck inequality) Show that the area of $\triangle ABC$ is bounded by $\dfrac{a^2 + b^2 + c^2}{4\sqrt{3}}$.

10. (For those who know about trigonometric functions)

    (Oppenheim-Mordell inequality) In the same setting as Problem 5, show that $PA \cdot PB \cdot PC \geq (PA' + PB')(PB' + PC')(PC' + PA')$.

11. (For those who know about complex numbers) For arbitrary points A, B, C, D in the plane, show that $AB \times CD + AD \times BC \geq AC \times BD$.