Berkeley Math Circle 2011

Problems with Projective Geometry

Let A, B, C and D be four points on a line l. The *cross ratio* of these points is

$$(AB;CD) = \frac{\frac{\overrightarrow{AC}}{\overrightarrow{AD}}}{\frac{\overrightarrow{BC}}{\overrightarrow{BD}}}.$$

Exercise 1.

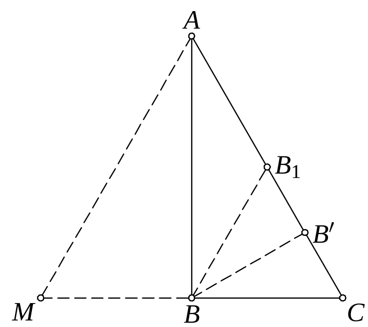
Triangle AMC is equilateral; B is the midpoint of MA, B' and B_1 are the feet of perpendicular and the midpoint of BC, respectively. Calculate

(a) $D_1 = (CB_1;B'A)$;

(b) $D_2 = (B'A;CB_1)$;

(c) $D_3 = (CB_1;AB')$;

(d) $D_1 \cdot D_3$;

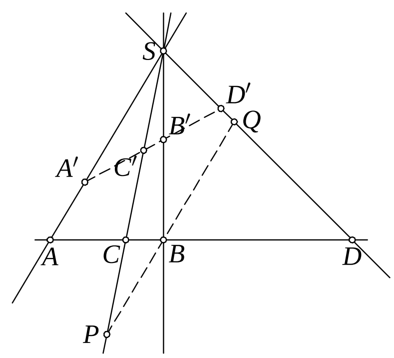


Theorem.

(i) Let A, B, C, D be four collinear points and S a point that doesn't belong to the line determined by the given four points. If P and Q are the points at which the line parallel to SA intersects SC and SD, then

$$(AB;CD) = \frac{\overrightarrow{BQ}}{\overrightarrow{BP}}.$$

(ii) If another line intersetcs SA, SB, SC, SD in the point A', B', C', D' then $(AB;CD) = (A'B';C'D')$. then DD' contains the point S.



(iii) If A, B, C, D belong to a line, and A', B', C', D' belong to the other line and if the following conditions are satisfied:

1° AA', BB' and CC' pass through the common point S;

2° $(AB;CD) = (A'B';C'D')$,

then DD' contains the point S.

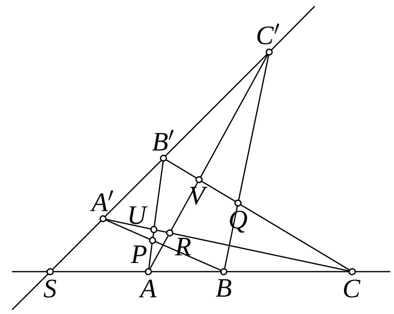
Exercise 2. Let ABC be isosceles rectangular triangle such that $\angle BAC = 90°$. Let A', B' and C' be the midpoints of BC, CA and AB, respectively. If M and N are intersection points of CC' with AA' and $A'B'$, respectively,

(a) Calculate $(CM;NC')$;

(b) Calculate $(A'C,A'M;A'N,A'C')$.

Exercise 3. Let ABC be rectangular triangle such that $\angle BAC = 90^\circ$. Let A' be the foot of the perpendicular from A to BC and P the foot of the perpendicular from A' to AC. Let Q be a point of the segment AB. If $\frac{\overrightarrow{QA}}{\overrightarrow{QB}} = k$, calculate $(A'C,A'A;A'P,A'Q)$.

Theorem. (Pappus theorem) Let (A,B,C) and (A',B',C') be two triples of collinear points. Denote by P, Q and R the following intersections of lines: $AB' \cap A'B$, $BC' \cap B'C$ and $AC' \cap A'C$. The points P, Q and R are colinear.



Proof. First, we will consider the case when the two given lines intersect. Let $S = AB \cap A'B'$, $U = AB' \cap A'C$, and $V = AC' \cap B'C$. Considering the following two lines: x determined by the points B', P, U, and A and y determined by S, B, C and A. We will apply part (ii) of the theorem from previous lecture to the lines x and y and the point A'. This gives us that $(B'P;UA) = (SB;CA)$. Similarly, observing the line y and the line z determined by B', Q, C and V, we can use the part (ii) of the same theorem (now we look at the point C) and we obtain that $(SB;CA) = (B'Q;CV)$. Hence, we conclude that $(B'P;UA) = (B'Q;CV)$. Now we apply part (iii) of the theorem from the previous lecture to conclude that P, Q and R are colinear. The case when $AB||A'B'$ is left as an exercise (whenever we used the part (ii) of the theorem in the preceding proof, you should use the part (i) of the same theorem for your exercise).

Definition. The point A, B, C, D are harmonic conjugated if $(AB;CD) = -1$.

Exercise 1. Given three colinear points A, B and C (such that C is between A and B), using only the ruler (without compass) construct a point D such that $(AB;CD) = -1$.
Hint. Choose an arbitrary point S and an arbitrary point I on SC. Let $B' = AI \cap SB$ and $A' = BI \cap SA$. Denote by C' the intersection point of $A'B'$ and SC, and denote by D the intersection point of AB and $A'B'$. Considering points I and S prove that $(AB;CD) = (B'A';C'D)$ and $(AB;CD) = (A'B';C'D)$...

Exercise 2. Let S and S' be the intersection points of internal and external bisector corresponding to the angle A of the triangle ABC with the line BC. Prove that $(S'S;AB) = -1$.
Remark The circle with diameter $S'S$ is called Apolonius circle of the triangle ABC. Needless to say, every triangle has three Apolonius circles.

Definition. If A is intersection point of the circle k and the line l, then the angle between k and l (usually denoted by $\angle(k,l)$ or $\measuredangle(k,l)$) is the non-obtuse angle between the line l and the tanget to circle k at the point A.

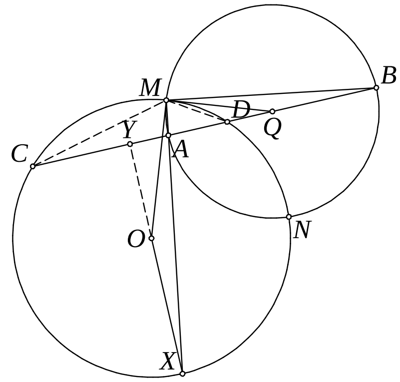
Exercise 1. Prove that the line l and the circle k are perpendicular (i.e. $\angle(k,l) = 90^\circ$) if and only if the center of k belongs to the line l.

Definition. If k_1 and k_2 are two circles intersecting at points A and B, the angle between those circles (usually denoted by $\angle(k_1,k_2)$ or $\measuredangle(k_1,k_2)$) is the non-obtuse angle between the tangents to those circles at the point A.

Exercise 2. Let k_1 and k_2 be two circles with centers O_1 and O_2. Let A be one of the intersection points of these circles. Prove that $\angle O_1AO_2 = \angle(k_1,k_2)$.

Definition. Points A and B are harmonic conjugated with the respect to circle k if $(AB;CD) = -1$, where C and D are intersection points of the line AB with the circle k.

Theorem. Points A and B are harmonic conjugated with the respect to circle k if and only if the circle l with diameter AB is perpendicular to k



Proof. Let O and Q denote the centers of the circles k and l (l is the circle with diameter AB). Denote by M and N the intersection points of k and l, and by C and D the intersection points of AB and k. Suppose that $k \perp l$. Let O and Q be the cdenters of k and l, respectively. Denote by X the intersection of MA with k and by Y the intersection of OX by CD. Then $\angle QMB = \angle OMX$ (angles with perpendicular rays). Since $\triangle OMX$ and QMB are isosceles, we conclude that $\angle OXM = \angle MBQ$ implying that $YXBM$ is cyclic which means that $\angle XYB = 90^\circ$. Since O is the center of k, this implies that OX is the bisector line of the segment CD. Thus X is the midpoint of the arc CD and hence MX is the bisector of $\angle CMD$, and the last exercise from the last lecture implies that $(CD'; AB) = -1$. The converse can be proved similarly.

Exercise 3. Let S and S' be the intersections of the internal and external bisectors of the angle A of $\triangle ABC$ with the line BC. Let T be the midpoint of SS'. Prove that $\angle TAB = \angle ACB$.

Theorem. Given a point A and the circle k wich doesn't contain A. The points conjugated to A with respect to k belongs to a line. This line is called the *polar line* of the point A with respect to k.
Proof. Let O be the center of the circle k, and let B be the arbitrary point conjugated to A with respect to k. Denote by P and Q intersection points of AO and k and let R be the intersection point of AO and l (l is the circle with diameter AB). If we prove that $(PQ; RA) = -1$, then we have proved the theorem, since this would imply that $BR \perp AO$ and R is fixed. To prove this notice that $l \perp k$. Since the circle with diameter PQ is perpendicular to l, we conclude that P and Q are conjugated with respect to l, implying the desired relation $(PQ; RA) = -1$.

Exercise 1. Construct the polar line to point A with respect to the circle k with the given center O using just a ruler.

Exercise 2. Given circle k and a point A in its exterior. If the center O of the circle k is given, prove that it is possible to construct a tangent from A to k using just a ruler.

Exercise 3. A circle k contains points B and C and intersects the sides AB and AC of $\triangle ABC$ at points Q and P. Let $BP \cap CQ = N$ and $BC \cap QP = M$. If U and V are intersection points of the line AN with the circle k, prove that MU and MV are tangent to k.