Berkeley Math Circle
Monthly Contest 7 – Solutions

1. Prove that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

for all positive integers n.

Solution.

$$\begin{aligned} &1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! \\ &= (2-1) \cdot 1! + (3-1) \cdot 2! + \cdots + [(n+1) - n] \cdot n! \\ &= (2 \cdot 1! - 1!) + (3 \cdot 2! - 2!) + \cdots + [(n+1) \cdot n! - n!] \\ &= (2! - 1!) + (3! - 2!) + \cdots + [(n+1)! - n!] \end{aligned}$$

When we add together the factorials in the last row, all terms cancel except for the $-1!$ at the beginning and the $(n+1)!$ at the end, so the sum is $(n+1)! - 1$.

Remark. This problem can also be solved using the method of mathematical induction.

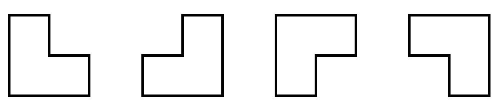
2. Do there exist four consecutive positive integers whose product is a perfect square?

Solution. The answer is no. If $x \geq 1$ is an integer,

$$\begin{aligned} &x(x+1)(x+2)(x+3) \\ &= [x(x+3)] \cdot [(x+1)(x+2)] \\ &= [x^2 + 3x] \cdot [x^2 + 3x + 2] \\ &= [(x^2 + 3x + 1) - 1] \cdot [(x^2 + 3x + 1) + 1] \\ &= (x^2 + 3x + 1)^2 - 1. \end{aligned}$$

Therefore the product of four consecutive positive integers is always one *less* than a square, and therefore cannot be a square since two positive squares cannot differ by 1.

3. Fix a positive integer n. Two players, Phil and Ellie, play the following game. First, Phil fills the squares of an $n \times n$ chessboard with nonnegative integers less than n. Then, Ellie chooses three squares making an L, as in any of the following pictures:



Ellie adds 1 to each of the three squares making the L, except that if the number n appears in a square, it is immediately replaced by 0. Ellie wins if, after modifying finitely many L's in this way, she can change all the numbers on the board into 0's; otherwise Phil wins.

Which player has a winning strategy if

(a) $n = 12$?

(b) $n = 2012$?

Solution. (a) Phil wins in this case. Consider the sum of all the numbers on the board. If Ellie adds 1 to three squares making an L, the sum increases by 3, and when a 12 is replaced by a 0, the sum decreases by 12. So the sum always increases and decreases by multiples of 3. So if Phil ensures that the initial sum is not a multiple of 3 (e.g. by putting a 1 in one square and 0's in all the others), then Ellie will never be able to make the sum 0.

(b) Ellie wins in this case. Consider four squares making a 2×2 square:

c	d
a	b

Suppose that Ellie chooses each of the L's $\{b, a, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ 671 times and the L $\{b, d, c\}$ 670 times. Then each of the squares b, c, d will be incremented exactly 2012 times, and so their values will remain the same after the operation. However, the square a will be incremented 2013 times; thus its value will be increased by 1 or, if it was initially 2011, replaced by 0. In a similar manner, Ellie can increment the value of any single square on the board and thus change all the numbers to 0's.

4. For a positive integer n, let $f(n)$ be the number of divisors of n which are perfect squares, and let $g(n)$ be the number of divisors of n which are perfect cubes. Determine whether there exists an integer n such that

$$\frac{f(n)}{g(n)} = 2012.$$

Solution. If n has the prime factorization

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

then the divisors of n which are perfect squares are the numbers of the form

$$p_1^{2a_1} p_2^{2a_2} \cdots p_k^{2a_k}$$

where each a_i is an integer such that $0 \le a_i \le e_i/2$. There are $\lfloor e_i/2 \rfloor + 1$ such integers, and we conclude that

$$f(n) = \left(\left\lfloor \frac{e_1}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{e_2}{2} \right\rfloor + 1\right) \cdots \left(\left\lfloor \frac{e_k}{2} \right\rfloor + 1\right).$$

By the same argument we find that

$$g(n) = \left(\left\lfloor \frac{e_1}{3} \right\rfloor + 1\right)\left(\left\lfloor \frac{e_2}{3} \right\rfloor + 1\right) \cdots \left(\left\lfloor \frac{e_k}{3} \right\rfloor + 1\right),$$

and thus the problem reduces to writing 2012 as a product of factors of the form

$$\frac{\lfloor x/2 \rfloor + 1}{\lfloor x/3 \rfloor + 1}$$

for various positive integers x. Taking $x = 2(2012 - 1)$ gives a fraction whose numerator is 2012, and this procedure can be applied iteratively:

$$2012 = \frac{2012}{1341} \cdot \frac{1341}{894} \cdot \frac{894}{596} \cdot \frac{596}{397} \cdot \frac{397}{265} \cdot \frac{265}{177} \cdot \frac{177}{118} \cdot \frac{118}{79} \cdot \frac{79}{53} \cdot \frac{53}{35} \cdot \frac{35}{23} \cdot \frac{23}{15} \cdot \frac{15}{10} \cdot \frac{10}{7} \cdot \frac{7}{5} \cdot \frac{5}{3} \cdot \frac{3}{2} \cdot \frac{2}{1} = \frac{f(n)}{g(n)}$$

where $n = 2^{4022} \cdot 3^{2680} \cdot 5^{1786} \cdot 7^{1190} \cdot 11^{792} \cdot 13^{528} \cdot 17^{352} \cdot 19^{234} \cdot 23^{156} \cdot 29^{104} \cdot 31^{68} \cdot 37^{44} \cdot 41^{28} \cdot 43^{18} \cdot 47^{12} \cdot 53^{8} \cdot 59^{4} \cdot 61^{2}$.

5. Let ABC be a triangle with incenter I. The circumcircle of $\triangle AIB$ meets the lines CA and CB again at P (different from A) and Q (different from B) respectively. Prove that A, B, P, and Q are (in some order) the vertices of a trapezoid.

Solution. Let ω and k be the circumcircles of $\triangle ABC$ and $\triangle AIB$ respectively, and let CI meet k again at M. Computing the angles of $\triangle AIM$ shows that it is isosceles with $MA = MI$. Similarly $MB = MI$. Therefore M is the center of k.

Assume for the moment that $CA \neq CB$. Consider reflection about the line CI, which fixes k and interchanges CA with CB. This reflection must take A to a point on both CB and k, but it cannot take A to B since $CA \neq CB$, so it must take A to Q. Similarly B reflects to P. Therefore $AQ \parallel BP$ since both are perpendicular to the line CI of reflection, and thus $APBQ$ is a trapezoid.

Now assume that $CA = CB$. Then CM is a diameter of ω, so $\angle CAM = 90$ and CA is tangent to k, which is inconsistent with the existence of P.

6. Determine all positive integers n such that there exist n distinct three-element subsets $A_1, A_2, \ldots, A_n$ of the set $\{1, 2, \ldots, n\}$ such that $|A_i \cap A_j| \neq 1$ for all i and j, $1 \le i < j \le n$.

Solution. The answer is all multiples of 4.

We will prove the following stronger statement by induction: Suppose that m distinct three-element subsets $A_1, A_2, \ldots, A_m$ of an n-element set are given, no two having a one-element intersection. Then m cannot exceed n, and m can only equal n if $4|n$.

We will induct (using "strong induction") on n, letting m vary freely subject to the condition that $m \geq n$. The cases $0 \leq n \leq 3$ are not hard to see.

Assume WLOG that $A_1 = \{1, 2, 3\}$. In our first case we assume that none of the other sets $A_2, \ldots, A_n$ intersect A_1. In this case we can remove A_1 and the three elements 1, 2, 3, thus decreasing n by 1 and m by 3 and getting a contradiction from the induction hypothesis.

So we assume that some other set does intersect A_1, WLOG $A_2 = \{1, 2, 4\}$ (the intersection must have size 2). Any set A_i that intersects A_1 does so in two elements and consequently intersects A_2, which in turn implies $|A_i \cap A_2| = 2$. Conversely any set that intersects A_2 must intersect A_1. These conditions are fulfilled only by two types of sets:

(a) $\{1, 2, x\}$, $x \notin \{1, 2, 3, 4\}$;

(b) $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

It is clear that sets in categories (a) and (b) cannot coexist. First assume that there are none in category (b), so all the sets containing 1 or 2 are (after renaming elements) $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \ldots, \{1, 2, k\}$ for some $k \geq 4$. No other set can contain any of the elements $1, 2, \ldots, k$, for then it would contain 1 or 2. Now we remove these k elements and the $k - 2$ sets containing them, and once again the induction hypothesis gives us a contradiction.

Finally we assume that one of the category (b) sets, say $\{1, 3, 4\}$, belongs to the collection of A_i's. Then it is easy to see that no other A_i, with the possible exception of $\{2, 3, 4\}$, can contain 1, 2, 3, or 4. We may assume that, in fact, $\{2, 3, 4\}$ is one of the given sets; including it will only make m larger. We then remove the four elements 1, 2, 3, 4 and the four sets composed of them. We deduce that $n - 4 = m - 4$ and $4|(n - 4)$, from which it follows that $n = m$ and $4|n$.

As our proof shows, a family of sets satisfying the conditions can be found for each multiple of 4 by partitioning the n elements into four-element blocks and including all the three-element subsets of each four-element block.

7. Consider the function

$$f(x) = \frac{(x-2)(x+1)(2x-1)}{x(x-1)}.$$

Suppose that u and v are real numbers such that

$$f(u) = f(v).$$

Suppose that u is rational. Prove that v is rational.

Solution. It is not hard to see that (for all real x except 0 and 1)

$$f(1-x) = -f(x) \quad \text{and} \quad f\left(\frac{1}{x}\right) = -f(x).$$

Therefore

$$f(x) = f\left(\frac{1}{1-x}\right) = f\left(1 - \frac{1}{x}\right).$$

If $u \in \mathbb{Q}$ is given, then $v = u$, $v = 1/(1-u)$, and $v = 1 - 1/u$ are three rational solutions to the equation $f(u) = f(v)$. Moreover, they are all distinct; setting any two equal yields a quadratic with the nonreal solutions $u = (1 \pm \sqrt{-3})/2$. However, the equation $f(u) = f(v)$, when expanded as a polynomial in v,

$$f(u) \cdot v(v-1) = (v-2)(v+1)(2v-1),$$

has degree 3. Therefore v cannot have any more than these three possible values.