Berkeley Math Circle
Monthly Contest 6 – Solutions

1. Is
$$\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\cdots+\frac{2011}{2012}$$
an integer? Prove your answer.

Solution. The answer is no. We present two solutions.

(1) We focus on the last term whose denominator is twice a prime number, namely $1993/1994$ with $1994 = 2 \cdot 997$. This is the only term whose denominator is divisible by the prime 997, and therefore the sum can be written as
$$\frac{A}{B}+\frac{1993}{1994},$$
where A/B is in lowest terms and B is not divisible by 997. Therefore the sum is not an integer, since two fractions whose sum is an integer must have the same denominator in lowest terms.

(2) We focus on the last term whose denominator is a power of 2, namely $1023/1024$ with $1024 = 2^{10}$. All the other terms have denominators with at most 9 factors of 2, so their least common denominator has at most 9 factors of 2. Therefore the sum can be expressed as
$$\frac{A}{B}+\frac{1023}{1024}$$
where A/B is in lowest terms and B is not divisible by 1024. Once again we have a contradiction with two fractions in lowest terms summing to an integer although their denominators are different.

2. At a market, a buyer and a seller each have four exotic coins. You are allowed to label each of the eight coins with any positive integer value in cents. The labeling is called n-*efficient* if for any integer k, $1 \le k \le n$, it is possible for the buyer and the seller to give each other some of their coins in such a way that, as a net result, the buyer has paid the seller k cents. Find the greatest positive integer n such that an n-efficient labeling of the coins exists.

Solution. The answer is 240.

To see that $n > 240$ is impossible, note that there are $2^8 = 256$ ways for the transaction to happen, since each coin either changes hands or does not change hands. However, the $2^4 = 16$ ways in which the buyer keeps all four of his coins clearly cannot allow the buyer to pay the seller a positive amount. Therefore there are at most 240 different amounts which the buyer can pay the seller.

To make these amounts the consecutive integers $1, 2, 3, \ldots, 240$, we assign values as follows:

Buyer: 16, 32, 64, 128
Seller: 1, 2, 4, 8.

Then the buyer can give the seller any multiple $16x$ of 16 cents up to 240 by writing x in binary as a sum of distinct powers of 2, and the seller can correspondingly give the buyer any amount from 0 to 15 cents in change.

3. Let $ABCD$ be a square in the coordinate plane such that A is on the x-axis and C is on the y-axis. Prove that one of the vertices B and D lies on the line $y = x$.

Solution. Assume without loss of generality that the vertices A, B, C, D are labeled in counterclockwise order. Let A and C have the coordinates $(a, 0)$ and $(0, c)$ respectively. The center M of the square is the midpoint of AC and therefore has the coordinates
$$\left(\frac{a}{2}, \frac{c}{2}\right).$$

To get from M to C, we can move upward $c/2$ units and then leftward $a/2$ units (of course, these are signed distances, indicating movement in the opposite directions if they are negative). Since B is the 90° clockwise rotation of C around M, we can get from M to B by moving rightward $c/2$ units and then upward $a/2$ units. This takes us from M to the point
$$\left(\frac{a}{2}+\frac{c}{2}, \frac{c}{2}+\frac{a}{2}\right),$$
which clearly lies on the line $y = x$.

4. Let a, b, c, x be real numbers such that

$$ax^2 - bx - c = bx^2 - cx - a = cx^2 - ax - b.$$

Prove that $a = b = c$.

Solution. Let $u = a - b$, $v = b - c$, and $w = c - a$. Then $u + v + w = 0$, and we would like to prove that $u = v = w = 0$. We have

$$\begin{aligned} ax^2 - bx - c &= bx^2 - cx - a \\ (a-b)x^2 + (-b+c)x + (-c+a) &= 0 \\ ux^2 - vx - w &= 0. \end{aligned} \tag{1}$$

In a similar manner, we get

$$vx^2 - wx - u = 0 \tag{2}$$

and

$$wx^2 - ux - v = 0. \tag{3}$$

Taking v times (1) minus u times (2) eliminates the x^2 term; we get

$$(v^2 - wu)x = u^2 - vw. \tag{4}$$

Letting $A = u^2 - vw$, $B = v^2 - wu$, $C = w^2 - uv$, we now have $Bx = A$, and analogously $Cx = B$ and $Ax = C$. We see that if any of A, B, and C are zero, then they all are. So we have two cases:

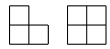
Case 1. $A = B = C = 0$, that is, $u^2 = vw$, $v^2 = wu$, and $w^2 = uv$. Clearly if one of u, v, and w is 0, then they all are and we are done. Otherwise, dividing the first equation by the second leads to $v^3 = w^3$ and $v = w$. Similarly $u = v = w$. But since $u + v + w = 0$, the only common value that u, v, and w can have is 0 and we are done.

Case 2. A, B, and C are all nonzero. Multiplying the three equations $Bx = A$, $Cx = B$, and $Ax = C$ together, we derive that $x^3 = 1$ so $x = 1$. Plugging $x = 1$ into the original equations gives

$$a - b - c = b - c - a = c - a - b,$$

so $a = b = c$ as desired.

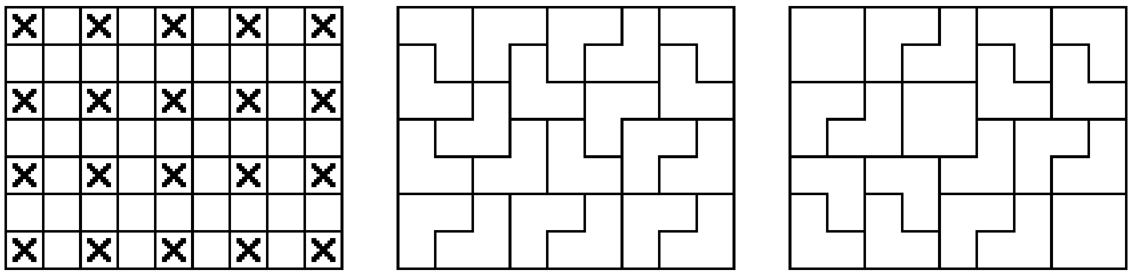
5. A 9×7 rectangle is tiled using only the two types of tiles below (the L-tromino and the 2×2 square):



They may be used in any orientation. Let s be the number of 2×2 squares in such a tiling; find all possible values of s.

Solution. Answer: 0 or 3. It is clear by considerations of area that the number of 2×2 squares must be a multiple of 3. Now we prove that a tiling with 6 or more 2×2 squares is impossible.

Draw $\times$'s in some of the cells of the 9×7 board as shown in the first of the following pictures.



Clearly, no tile of either of the two given shapes can cover more than one $\times$. So at least 20 tiles must be used. But if there are $s \geq 6$ of the 2×2 tiles, the number of L-tiles is

$$\frac{63 - 4s}{3}$$

and the total number of tiles is

$$s + \frac{63 - 4s}{3} = \frac{63 - s}{3} \leq \frac{63 - 6}{3} = 19,$$

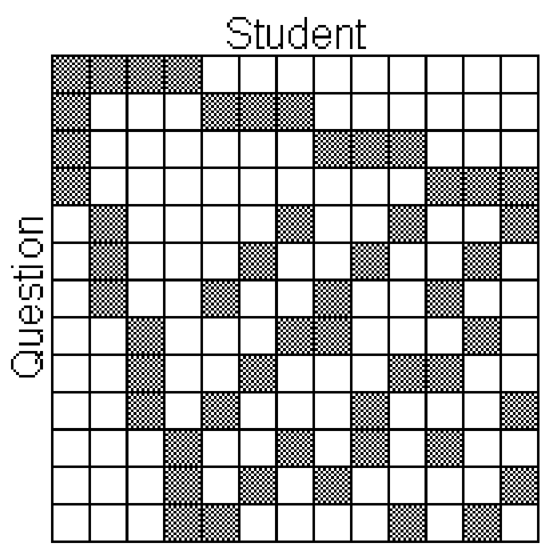
a contradiction.

The remaining two figures above show how the tiling may be accomplished with zero and three 2×2 squares respectively.

6. On a quiz, every question is solved by exactly four students, every pair of questions is solved by exactly one student, and none of the students solved all of the questions. Find the maximum possible number of questions on the quiz.

Solution. Number the students and the questions so that student S_1 solved questions $Q_1, \ldots, Q_k$ but not $Q_{k+1}, \ldots, Q_n$. Assume first that $k > 4$. Since Q_{k+1} is solved by exactly 4 students, while Q_i and Q_{k+1} are solved by exactly one student for each $1 \leq i \leq k$, there must be another student who solved two of the questions $Q_1, \ldots, Q_k$ besides S_1, a contradiction. Therefore each student solved at most four questions. We conclude that there are at most 13 questions—Q_1 plus at most 3 other questions solved by each of the 4 students who solved Q_1.

The following example shows that a quiz with 13 questions is possible.



7. In acute triangle ABC, the exterior angle bisector of $\angle BAC$ meets ray BC at D. Let M be the midpoint of side BC. Points E and F lie on the line AD such that $ME \perp AD$ and $MF \perp BC$. Prove that

$$BC^2 = 4AE \cdot DF.$$

Solution. Because $DM^2 = DE \cdot DF$, we can forget point F and replace the condition to be proved by

$$DE \cdot BC^2 \stackrel{?}{=} 4AE \cdot DM^2.$$

Let AG be the internal bisector of angle BAC (G is on BC). Since the internal and external angle bisectors are perpendicular, we have $\triangle DAG \sim \triangle DEM$ and

$$\frac{DA}{DE} = \frac{DG}{DM}$$
$$\frac{AE}{DE} = \frac{GM}{DM}.$$

Therefore we can forget point E and replace the condition to be proved by

$$BC^2 \stackrel{?}{=} 4GM \cdot DM.$$

We have

$$\begin{aligned} 4GM \cdot DM &= 4\left(\frac{GB - CG}{2}\right)\left(\frac{DB + DC}{2}\right) \\ &= (GB - CG)(DB + DC) \end{aligned}$$

and

$$BC^2 = (GB + CG)(DB - DC).$$

Expanding out the equality of these two expressions gives $CG \cdot DB \stackrel{?}{=} GB \cdot DC$, which is clear from the Angle Bisector Theorem

$$\frac{AB}{AC} = \frac{GB}{CG} = \frac{DB}{DC}.$$