Berkeley Math Circle
Monthly Contest 5
Due February 7, 2012

Instructions

This contest consists of 7 problems, some of which are easier than the others. Problems 1–4 comprise the Beginner Contest (for grades 8 and below) and Problems 3–7 comprise the Advanced Contest (for grades 9–12). Every problem is worth 7 points. Please write your solution to every problem on a separate sheet of paper, and on top of each sheet include your name, grade, school, and BMC level, as well as the problem number and the contest number. Thus, the header on each sheet should look something like:

Solution to Problem 3 of BMC Monthly Contest 5
by Bart Simpson
in grade 7, BMC Intermediate
from Springfield Middle School, Springfield

If you submit more than one sheet for a specific problem, please, staple the sheets together to avoid getting them confused with someone else's solution. Please, do NOT staple together solutions to DIFFERENT problems, as they will be graded separately.

Carefully justify your answers to avoid losing points. Include all relevant explanations in words and all intermediate calculations. Answers without justification will receive no credit. However, good reasoning with minor calculational errors may receive a lot of points. Thus, submit solutions to as many problems as you can since partial credits will be awarded for sufficient progress on any particular problem.

Remember that you are NOT ALLOWED to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, or other means of communication. You can consult any book that you wish. For more on the contest rules, please, check the BMC website at http://mathcircle.berkeley.edu.

Enjoy solving these problems and good luck!

Problems

1. Each vertex of a cube is labeled with an integer. Prove that there exist four coplanar vertices the sum of whose numbers is even.
2. A 2012×2012 table is to be filled with integers in such a way that each of the 4026 rows, columns, and main diagonals has a different sum. What is the smallest number of distinct values that must be used in the table?
3. Let x, y, and z be real numbers.
   (a) Prove that
   $$x + y > |x - y|$$
   if and only if x and y are both positive.
   (b) Find an inequality involving the variables x, y, and z, using only the operations of addition, subtraction, multiplication and absolute value, that is true if and only if x, y, and z are all positive.
4. The tangents at A and B to the circumcircle of an acute triangle ABC intersect at T. Point D lies on line BC such that $DA = DC$. Prove that $TD \parallel AC$.
5. A grasshopper begins at the vertex $(1, 0)$ of an infinite grid with origin $O(0, 0)$. It can jump from a vertex A to any vertex B such that $\triangle OAB$ has area $1/2$.
   (a) Find all points (x, y) of the grid which the grasshopper can reach.
   (b) Prove that if the grasshopper can reach a point (x, y), then it can reach it in at most $|y| + 2$ jumps, starting from $(1, 0)$.
6. Determine whether there exist two distinct finite subsets A and B of the reals such that for every polynomial f of degree 2012 with real coefficients,
   $$\sum_{x \in A} f(x) = \sum_{x \in B} f(x).$$
7. Find all primes p such that there exist integers a, b, c, and k satisfying the equations
   $$a^2 + b^2 + c^2 = p$$
   $$a^4 + b^4 + c^4 = kp.$$