Berkeley Math Circle
Monthly Contest 4 – Solutions

1. On a given street, there are n houses numbered from 1 to n. Let a_i ($1 \le i \le n$) be the number of people living in the house numbered i, and let b_i ($i \ge 1$) be the number of houses on the street in which at least i people live. Prove that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + b_3 + \cdots .$$

Solution. Let us number the people in each house from 1 up to the total number of people living there. Then for each $k \ge 1$, the label k is assigned as often as there is a house with k or more people; thus there are b_k people labeled k. The quantity

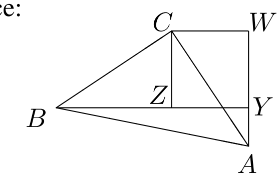
$$b_1 + b_2 + b_3 + \cdots$$

thus represents the total number of people labeled with some positive integer; this is the total number of people in all the houses, $a_1 + \cdots + a_n$.

2. Let ABC be an isosceles right triangle with $\angle C = 90°$. Let X be a point on the line segment AC, and let Y and Z be, respectively, the feet of the perpendiculars from A and C to the line BX. Prove that

$$BZ = YZ + AY.$$

Solution. Draw the perpendicular CW from C to line AY. We get $\triangle CBZ \cong \triangle CAW$ by AAS since:



- $\angle BCZ = \angle ACW$ (this comes from the right angles BCA and ZCW).
- $\angle BZC = \angle AWC$ (both $90°$)
- $BC = AC$.

Quadrilateral $CZYW$ must be a square, since it already has four right angles and we just proved that $CZ = CW$. Hence

$$BZ = AW = YW + AY = YZ + AY.$$

Remark. This problem can also be solved readily using coordinates so that lines BY, AY, and CZ are parallel to the axes.

3. Determine, with proof, whether the following statement is true or false: Out of any six natural numbers, one can find either three which are pairwise relatively prime or three whose greatest common divisor is greater than 1.

Solution. The statement is false. To avoid having three pairwise relatively prime numbers, we should give each pair of numbers a prime factor in common; to avoid having three numbers with a nontrivial gcd, we should not use any factor to cover more than two numbers. Utilizing the first fifteen primes for the $\binom{6}{2}$ pairs, we arrive at the numbers:

$$\begin{gathered} 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \\ 2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \\ 3 \cdot 13 \cdot 29 \cdot 31 \cdot 37 \\ 5 \cdot 17 \cdot 29 \cdot 41 \cdot 43 \\ 7 \cdot 19 \cdot 31 \cdot 41 \cdot 47 \\ 11 \cdot 23 \cdot 37 \cdot 43 \cdot 47 \end{gathered}$$

In fact, no *pair* of these numbers are relatively prime, while no three have a common factor.

4. A $2 \times n$ grid has a light bulb in each square. Each bulb has a switch that flips the state of its corresponding bulb as well as all (horizontally or vertically) adjacent bulbs. Determine whether it is always possible, regardless of the initial state of the bulbs, to turn all the bulbs off if

   (a) $n = 2011$

   (b) $n = 2012$.

Solution. The answer to both questions is no. Call a lamp "hot" if it is in one of the squares marked $\times$ in the following two patterns:

For 2×2011:

$\times$		$\times$		$\times$	$\cdots$	$\times$		$\times$
$\times$		$\times$		$\times$	$\cdots$	$\times$		$\times$

For 2×2012:

	$\times$	$\times$	$\times$		$\times$	$\times$	$\times$	$\cdots$		$\times$	$\times$	$\times$
		$\times$				$\times$		$\cdots$			$\times$	

It is not hard to see that any flip of a switch affects precisely zero, two, or four hot bulbs, which keeps the parity of the number of *lit* hot bulbs the same. Thus if we begin with only one lit hot bulb, the number of lit hot bulbs will always be odd and we can never turn all the bulbs off.

5. Determine all pairs (n, k) of integers such that $0 < k < n$ and

$$\binom{n}{k-1} + \binom{n}{k+1} = 2\binom{n}{k}.$$

Solution. In the factorial form,

$$\frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{2 \cdot n!}{k!(n-k)!}$$

we multiply through by $(k+1)!(n-k+1)!$ to clear the fractions and then divide through by $n!$:

$$k(k+1) + (n-k)(n-k+1) = 2(k+1)(n-k+1).$$

To decrease the number of terms, we let $k + 1 = a$ and $n - k + 1 = b$:

$$\begin{aligned} (a-1)a + (b-1)b &= 2ab \\ a^2 - 2ab + b^2 &= a + b \\ (a-b)^2 &= a + b \end{aligned}$$

If we let $a - b = c$, then $a + b = c^2$ and we get

$$a = \frac{c^2 + c}{2} \quad \text{and} \quad b = \frac{c^2 - c}{2}$$

Here any integer value of c will yield nonnegative integer values of a and b; however, the condition $0 < k < n$ requires that $a = k + 1$ and $b = n - k + 1$ are each at least 2. Hence the values $c = -2, -1, 0, 1, 2$ are excluded, while every $c \leq -3$ and every $c \geq 3$ will yield permissible values for

$$k = a - 1 = \frac{c^2 + c - 2}{2}$$

and

$$n = a + b - 2 = c^2 - 2$$

which satisfy the equation.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy + z) = f(x)f(y) + f(z)$$

for all real numbers x, y, and z.

Solution. Answer: $f(x) = 0$ or $f(x) = x$. Both of these trivially satisfy the equation.

We plug in $x = y = z = 0$ to get

$$\begin{aligned} f(0) &= f(0)^2 + f(0) \\ 0 &= f(0)^2 \\ 0 &= f(0). \end{aligned}$$

Then we plug in $x = y = 1$, $z = 0$ to get

$$f(1) = f(1)^2 + f(0)$$
$$0 = f(1)^2 - f(1)$$

so $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, we can immediately plug in $y = 1$ and $z = 0$ to conclude that

$$f(x) = f(x) \cdot 0 + 0 = 0$$

for all x.

If $f(1) = 1$, then we can derive that f has two simple properties: the *addition property* from plugging in $y = 1$,

$$f(x + z) = f(x) + f(z)$$

and the *multiplication property* from plugging in $z = 0$,

$$f(xy) = f(x)f(y).$$

Plugging $z = -x$ into the addition property tells us that $f(-x) = -f(x)$, i.e. f is an odd function. Also, the addition property lends itself to use for an induction, starting at $f(1) = 1$, that proves $f(n) = n$ for all positive integers n.

Suppose that f is *not* the identity function. Then there is a number x such that $f(x) \neq x$. Changing x to $-x$ if necessary, we can assume that $f(x) < x$. We can then double x repeatedly, which will also double $f(x)$ by the addition property, until the difference $x - f(x)$ becomes at least 2. Then there is an integer n between $f(x)$ and x:

$$f(x) < n < x.$$

We note that $x - n$ is positive, while $f(x) - n = f(x - n)$ is negative. Thus we have a number $u > 0$ such that $f(u) < 0$. This is impossible by the multiplication property, since for $u > 0$,

$$f(u) = f(\sqrt{u} \cdot \sqrt{u}) = f(\sqrt{u}) \cdot f(\sqrt{u}) \geq 0.$$

7. Let ABC be a triangle. The incircle, centered at I, touches side BC at D. Let M be the midpoint of the altitude from A to BC. Lines MI and BC meet at E. Prove that $BD = CE$.

   Solution. Let E' be the point on side BC such that $BD = E'C$. Then it is well known that E' is the tangency point of the A-excircle Ω of $\triangle ABC$ with BC. Let H be the foot of the altitude from A to BC, and let K be the point on the incircle ω opposite D.

   Circles ω and Ω are homothetic about A with K and E' corresponding points, so A, K, and E' are collinear. Thus triangles AHE' and KDE' are homothetic; M and I are midpoints of corresponding sides, so they are collinear with E'. So $E' = E$, as desired.