Berkeley Math Circle
Monthly Contest 3 – Solutions

1. Let a and b be integers such that
$$|a+b| > |1+ab|.$$
Prove that $ab = 0$.

Solution. Notice that replacing both a and b by their negatives does not change either side of the given inequality. Therefore, we may assume that $a + b \geq 0$. We now have $a + b > |1 + ab|$, so

$$\begin{aligned} a+b > 1+ab \quad &\text{and} \quad a+b > -1-ab \\ ab-a-b+1 < 0 \quad &\text{and} \quad ab+a+b+1 > 0 \\ (a-1)(b-1) < 0 \quad &\text{and} \quad (a+1)(b+1) > 0. \end{aligned}$$

If a and b are nonzero, then $a-1$ and $a+1$ are both nonnegative or both nonpositive, and similarly for $b-1$ and $b+1$. So $(a-1)(b-1)$ and $(a+1)(b+1)$ are both nonnegative or both nonpositive, which contradicts the information above.

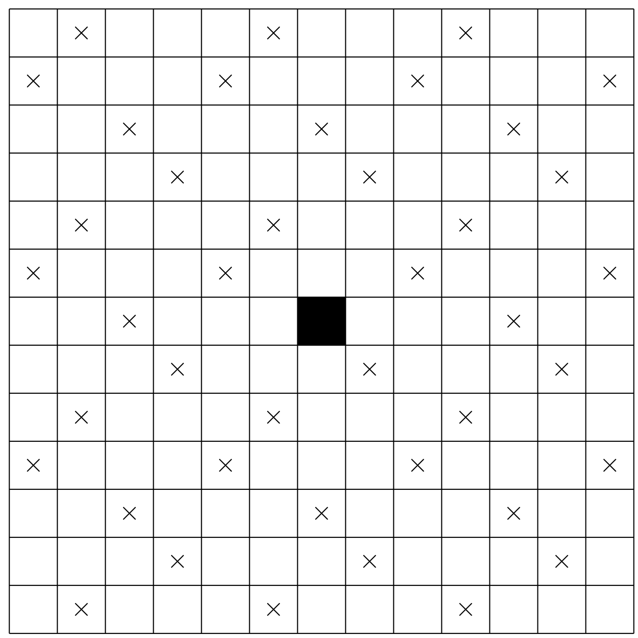
Remark. A longer, but easier to find, proof may be constructed by separately considering the four cases according to whether a and b are positive or negative.

2. Let p be a prime number. Find all possible values of the remainder when $p^2 - 1$ is divided by 12.

Solution. The answers are 3, 8, and 0.
It is clear that $p = 2$ gives 3, $p = 3$ gives 8, and $p = 5$ gives 0. We claim that all primes $p \geq 5$ give the remainder 0 as well, i.e. that $p^2 - 1$ is divisible by 12 for these p.
We factor:
$$p^2 - 1 = (p+1)(p-1)$$
Since $p \neq 2$, p is odd and so $p+1$ and $p-1$ are both even. This gives two factors of 2 in $p^2 - 1$. Moreover, one of the three consecutive integers $p-1$, p, $p+1$ is divisible by 3, and since $p \neq 3$, it is not p. So either $p-1$ or $p+1$ has a factor of 3, and so does $p^2 - 1$. Thus $p^2 - 1$ is divisible by $2 \cdot 2 \cdot 3 = 12$.

3. We are given a 13×13 chessboard. Determine whether it is possible to place nonoverlapping 1×4 rectangular tiles on the board so as to cover every square but the central one.

Solution. It is impossible. In the diagram at right, each of the 42 tiles used to cover the 168 vacant squares must cover exactly one $\times$. But there are only 41 $\times$'s!



4. Let $ABCD$ be a parallelogram. Suppose that the circumcenter of $\triangle ABC$ lies on diagonal BD. Prove that $ABCD$ is either a rectangle or a rhombus (or both).

Solution. To get a conclusion of the appropriate type (a rectangle OR a rhombus), we must divide up the problem into two cases. Here is one way of accomplishing this:

Case 1. The circumcenter O of ABC is the *center* of $ABCD$, the common midpoint of diagonals AC and BD. Then since radii OA and OB are equal, we get $AC = 2OA = 2OB = BD$. Thus $ABCD$ is a parallelogram whose diagonals are congruent, i.e. a rectangle.

Case 2. The circumcenter O of ABC does not coincide with the midpoint M of AC and BD. Then since O is on the perpendicular bisector of AC, we have $OM \perp AC$. But O and M are both on line BD, so $BD \perp AC$. Thus $ABCD$ is a parallelogram whose diagonals are perpendicular, i.e. a rhombus.

5. Let $\ominus$ be an operation on the set of real numbers such that

$$(x \ominus y) + (y \ominus z) + (z \ominus x) = 0$$

for all real x, y, and z. Prove that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$x \ominus y = f(x) - f(y)$$

for all real x and y.

Solution. First we plug in $x = y = z = 0$ to get

$$(0 \ominus 0) + (0 \ominus 0) + (0 \ominus 0) = 0,$$

that is, $0 \ominus 0 = 0$. Then we plug $y = z = 0$, keeping x undetermined, into the original equation to get

$$(x \ominus 0) + 0 + (0 \ominus x) + 0 = 0.$$

So $0 \ominus x = -(x \ominus 0)$. Define $g(x) = x \ominus 0$. Plugging $z = 0$ into the original equation gives

$$\begin{aligned}(x \ominus y) + (y \ominus 0) + (0 \ominus x) &= 0 \\ (x \ominus y) + (y \ominus 0) - (x \ominus 0) &= 0 \\ (x \ominus y) &= (x \ominus 0) - (y \ominus 0) \\ &= g(x) - g(y),\end{aligned}$$

as desired.

6. Let N be a positive integer such that N is divisible by 81 and the number formed by reversing the digits of N is also divisible by 81. Prove that the sum of the digits of N is divisible by 81.

Solution. We begin with a lemma.

Lemma. For all $k \geq 0$, $10^k \equiv 1 + 9k \mod 81$.

Proof. Binomial theorem:

$$10^k = (1+9)^k = 1^k + \binom{k}{1} \cdot 1^{k-1} \cdot 9 + \text{terms divisible by } 9^2. \qquad \square$$

We now write N in terms of its digits as

$$\begin{aligned} N &= a_0 + 10a_1 + \cdots + 10^n a_n \\ &\equiv (a_0 + 9 \cdot 0 \cdot a_0) + (a_1 + 9 \cdot 1 \cdot a_1) + \cdots + (a_n + 9 \cdot n \cdot a_n) \\ &\equiv a_0 + a_1 + \cdots + a_n + 9(0a_0 + 1a_1 + 2a_2 + \cdots + na_n) \bmod 81. \end{aligned}$$

Correspondingly, the number formed by reversing the digits of N is

$$a_n + 10a_{n-1} + \cdots + 10^n a_0 \equiv a_0 + a_1 + \cdots + a_n + 9(na_0 + (n-1)a_1 + \cdots + 0a_n) \bmod 81.$$

If we add these two numbers, we get that 81 divides

$$2(a_0 + \cdots + a_n) + 9(na_0 + na_1 + \cdots + na_n) = (9n+2)(a_0 + \cdots + a_n).$$

Since $9n + 2$ is not divisible by 3, the conclusion follows.

7. Let k be a positive integer, and let $(a_1, a_2, \ldots, a_{2k})$ and $(b_1, b_2, \ldots, b_{2k})$ be two sequences of real numbers such that $1/2 \leq a_1 \leq \cdots \leq a_{2k}$ and $1/2 \leq b_1 \leq \cdots \leq b_{2k}$. Let M and m be the maximum and minimum respectively of

$$(a_1 + c_1)(a_2 + c_2) \cdots (a_{2k} + c_{2k}) \tag{1}$$

as $(c_1, \ldots, c_{2k})$ ranges through all possible permutations of $(b_1, \ldots, b_k)$. Prove that

$$M - m \geq k(a_k - a_{k+1})(b_k - b_{k+1}).$$

Solution. Let $c_i = b_{\sigma(i)}$ where σ is a permutation of the numbers 1 through $2k$ that will vary. We begin with σ being the numbers 1 through $2k$ written in order,

$$1 \quad 2 \quad \cdots \quad 2k,$$

so $c_i = b_i$ for each i. We then switch 1 with 2, 1 with 3, 1 with 4, and so on, until 1 arrives to the right of $2k$. Then we switch 2 with 3, 2 with 4, etc., until 2 is just to the left of 1. We continue in this way until the order of all the numbers $\sigma(i)$ is reversed. Each of these switches changes the permutation σ and therefore the product (1). We claim that after all of them have been performed, the value of (1) is increased by at least $k(a_k - a_{k+1})(b_k - b_{k+1})$.

Lemma. No switch decreases the value of (1).

Proof. Each switch interchanges the values of two consecutive c_i's, replacing two factors $(a_i + c_i)(a_{i+1} + c_{i+1})$ of the product by $(a_i + c_{i+1})(a_{i+1} + c_i)$ and increasing the value by a positive constant times

$$(a_i + c_{i+1})(a_{i+1} + c_i) - (a_i + c_i)(a_{i+1} + c_{i+1}) = (a_{i+1} - a_i)(c_{i+1} - c_i).$$

The factor $(a_{i+1} - a_i)$ is obviously nonnegative, and the factor $(c_{i+1} - c_i)$ is nonnegative also due to the way we performed the switches: a smaller number always moves to the right while a larger number moves to the left, so $c_i = b_m$ and $c_{i+1} = b_n$ with $m < n$. $\square$

Lemma. At least k switches occur between c_k and c_{k+1} in which initially $\sigma(k) \leq k$ and $\sigma(k+1) > k$ (and subsequently these inequalities are reversed).

Proof. Let $m(\sigma)$ be the number of indices i such that $i \leq k$ and $\sigma(i) \leq k$. Initially, $m(\sigma) = k$ and ultimately $m(\sigma) = 0$. Only switches of the type described alter $m(\sigma)$, and only by 1, so there must be at least k of them (in fact there are exactly k). $\square$

We complete the proof by showing that each switch of the type described in the above lemma increases the value of (1) by at least $(a_k - a_{k+1})(b_k - b_{k+1})$. By the calculation in Lemma 7, the product of the two factors altered by the switch is increased by

$$(a_{i+1} - a_i)(c_{i+1} - c_i) \geq (a_{k+1} - a_k)(b_k - b_{k+1});$$

the remaining factors all have the form

$$a_i + c_i \geq \frac{1}{2} + \frac{1}{2} = 1,$$

so the increase in the product is at least $(a_k - a_{k+1})(b_k - b_{k+1})$.