Berkeley Math Circle
Monthly Contest 2 – Solutions

1. Find the number of multiples of 3 which have six digits, none of which is greater than 5.

   Solution. The first digit can be any number from 1 to 5, making 5 possibilities. Each of the succeeding digits, from the ten-thousands digit to the tens digit, can be any of the six digits from 0 to 5. Finally, we claim that there are exactly two possibilities for the last digit. Given the first five digits, if we append the digits 0, 1, and 2 in turn, we get three consecutive integers, exactly one of which is a multiple of 3. The same happens when we add the digits 3, 4, and 5.

   Thus the total number of multiples of 3 is $5 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 2 = 12960$.

2. On an infinite chessboard, two squares are said to *touch* if they share at least one vertex and they are not the same square. Suppose that the squares are colored black and white such that

   - there is at least one square of each color;
   - each black square touches exactly m black squares;
   - each white square touches exactly n white squares

   where m and n are integers. Must m and n be equal?

   Solution. The answer is no. There are many tilings to demonstrate this; one of the simplest is to divide the board into horizontal stripes and color every third stripe black. In this tiling, $m = 2$ and $n = 5$.

3. Is there an integer x such that

   $$2010 + 2009x + 2008x^2 + 2007x^3 + \cdots + 2x^{2008} + x^{2009} = 0?$$

   Solution. The answer is no.

   It is clear that if x is positive, the left side is positive, and if $x = 0$, the left side is 2010. If $x = -1$, the left side is

   $$(2010 - 2009) + (2008 - 2007) + \cdots + (2 - 1) = 1 + 1 + \cdots + 1,$$

   likewise a positive number.

   If $x \leq -2$, we claim that the left side is negative. Pair the terms again and factor:

   $$(2010 + 2009x) + x^2(2008 + 2007x) + \cdots + x^{2006}(4 + 3x) + x^{2008}(2 + x).$$

   Each of the binomials in parentheses has the form $(a + 1) + ax$, with $a \geq 1$, and its value is at most

   $$(a + 1) + a(-2) = 1 - a \leq 0.$$

   Moreover, only the last binomial, for $a = 1$, is capable of equaling 0; the others are strictly negative. The coefficients 1, x^2, x^4, etc. of these binomials are of course positive, yielding a negative sum.

4. Let $ABCD$ be a convex quadrilateral such that $\angle ABD = \angle ACD$. Prove that $ABCD$ can be inscribed in a circle.

   Solution. There are many ways to structure the proof. The following method seems to have minimal logical difficulties.

   Because points A, B, and C are not collinear, we can draw the circumscribed circle ω of $\triangle ABC$. The arc AC of ω, not containing B, is intercepted by inscribed angle ABC and thus has measure $2\angle ABC$. On this arc we may find a point E such that $\overset{\frown}{AE}$ has the smaller measure $2\angle ABD$. Then angles ABD and ABE have the same measure and orientation, so E is on BD; also, angles ACD and ACE have the same measure and orientation, so E is on CD. Since lines BD and CD have only one point in common, $D = E$ and thus D lies on the circle.

5. Let $n > 3$ be a positive integer. Define an integer k to be *snug* if $1 \leq k < n$ and

   $$\gcd(k, n) = \gcd(k + 1, n).$$

   Prove that the product of all snug integers is congruent to 1 modulo n.

   Remark. If there are no snug integers, their product is vacuously declared to equal 1.

Solution. Let k be a snug integer. Note that any factor that divides n, k, and $k+1$ must also divide $(k+1)-k=1$, so

$$\gcd(k,n)=\gcd(k+1,n)=1.$$

In particular, k has a multiplicative inverse h mod n (we can choose h such that $0<h<n$). We claim that h is also snug. Clearly $\gcd(h,n)=1$; note that

$$(h+1)\cdot k = hk+k \equiv k+1 \bmod n;$$

since k and $k+1$ are invertible mod n, so is $h+1$.

Thus we can pair up snug residues mod n into pairs with product 1, unless there is a snug k that is its own multiplicative inverse. We claim that there is no such k except possibly $k=1$. Indeed, if $k^2\equiv 1 \bmod n$, then n divides

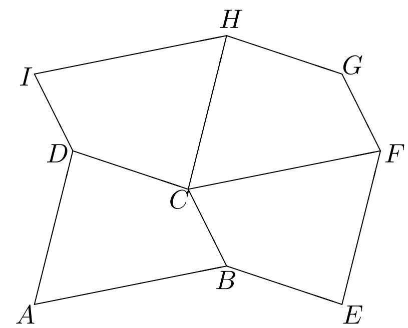
$$k^2-1=(k+1)(k-1).$$

Since k is snug, n is relatively prime to $k+1$ and hence divides $k-1$, implying that $k=1$.

6. Let $ABCD$ be a convex quadrilateral. Suppose that the area of $ABCD$ is equal to

$$\frac{AB+CD}{2}\cdot\frac{AD+BC}{2}.$$

Prove that $ABCD$ is a rectangle.



Solution. In the picture shown, we have replicated quadrilateral $ABCD$ four times, rotating by $180°$ successively about the midpoints of BC, CF, and CH. Because the angles of the quadrilateral sum to $360°$, the figure "closes up" so that the final quadrilateral has CD as a side and would, if further rotated $180°$ about CD, coincide with $ABCD$.

We note that AD and EF are parallel and congruent, as are FG and DI; thus triangles ADI and EFG are translations of one another, and AI is parallel and and congruent to IG. The same can be said for triangles ABE and IHG. So

$$4\cdot \text{Area}\, ABCD = \text{Area}\, ABEFGHID = \text{Area}\, ABEGHI = \text{Area}\, AEGI.$$

Because $AEGI$ is a parallelogram, with base AE and height at most AI, we can continue:

$$\text{Area}\, AEGI \le AE\cdot AI \le (AB+BE)(AD+DI) = (AB+CD)(AD+BC) = 4\,\text{Area}\, ABCD.$$

All these inequalities must be equalities. So A, B, E are collinear (i.e. $AB \parallel CD$); A, D, I are collinear (i.e. $AD\parallel BC$); and angle EAI, which is also angle BAD, equals $90°$. From this we conclude that $ABCD$ is a rectangle.

7. Let N be a positive integer. Define a sequence a_n, $n\ge 0$, by

$$a_0=0, \qquad a_1=1, \qquad a_{n+1}+a_{n-1}=a_n\left(2-\frac{1}{N}\right) \quad (n\ge 1).$$

Prove that $a_n<\sqrt{N+1}$ for all $n\ge 0$.

Solution.

Lemma. For all $n\ge 0$, a_n and a_{n+1} are related by a quadratic equation, namely

$$a_{n+1}^2+a_n^2-\left(2-\frac{1}{N}\right)a_na_{n+1}=1. \tag{1}$$

Proof. By induction. The case $n=0$ is obvious. Suppose that for some $n\ge 0$, (1) holds. Then

$$\begin{aligned}
a_{n+2}^2+a_{n+1}^2-\left(2-\frac{1}{N}\right)a_{n+1}a_{n+2} &= a_{n+1}^2-\left[\left(2-\frac{1}{N}\right)a_{n+1}-a_{n+2}\right]a_{n+2}\\
&= a_{n+1}^2-a_na_{n+2}\\
&= a_{n+1}^2-\left[\left(2-\frac{1}{N}\right)a_{n+1}-a_n\right]a_n\\
&= a_{n+1}^2+a_n^2-\left(2-\frac{1}{N}\right)a_na_{n+1}\\
&= 1.
\end{aligned}$$

□

If a_n is treated as a constant, equation (1), the quadratic

$$a_{n+1}^2 - \left(2 - \frac{1}{N}\right) a_n a_{n+1} + (a_n^2 - 1) = 0$$

must have at least one real root a_{n+1}, so its discriminant is nonnegative:

$$\begin{aligned} \left(2 - \frac{1}{N}\right)^2 a_n^2 &\geq 4 \cdot 1 \cdot (a_n^2 - 1) \\ \left(\left(2 - \frac{1}{N}\right)^2 - 4\right) a_n^2 &\geq -4 \\ \left(\frac{-4N+1}{N^2}\right) a_n^2 &\geq -4 \\ a_n^2 &\leq \frac{4N^2}{4N-1}. \end{aligned}$$

Thus it suffices to show that

$$\begin{aligned} \frac{4N^2}{4N-1} &< N + 1 \\ 4N^2 &< (N+1)(4N-1) = 4N^2 + 3N - 1 \\ 1 &< 3N, \end{aligned}$$

which is true.