Berkeley Math Circle
Monthly Contest 1 – Solutions

1. Let x be an odd positive integer other than 1. Prove that one can find positive integers y and z such that

$$x^2 + y^2 = z^2.$$

Solution. Let

$$y = \frac{x^2 - 1}{2} \qquad \text{and} \qquad z = \frac{x^2 + 1}{2}.$$

Because x is odd, $x^2 - 1$ and $x^2 + 1$ are both even and therefore y and z are integers. Moreover, because x is more than 1, $x^2 - 1$ and $x^2 + 1$ are more than 0 and thus y and z are positive. Finally, the desired equation $x^2 + y^2 = z^2$ is equivalent to

$$\begin{aligned} x^2 + \left(\frac{x^2-1}{2}\right)^2 &= \left(\frac{x^2+1}{2}\right)^2 \\ x^2 + \frac{(x^2-1)^2}{4} &= \frac{(x^2+1)^2}{4} \\ 4x^2 + (x^2-1)^2 &= (x^2+1)^2 \\ 4x^2 + x^4 - 2x^2 + 1 &= x^4 + 2x^2 + 1, \end{aligned}$$

which is true.

2. A finite number of points are drawn in the plane. Prove that one can select two of them, A and B, such that:

   (a) A and B are not the same point.

   (b) No drawn point, other than A itself, is closer to A than B is.

   (c) No drawn point, other than B itself, is closer to B than A is.

   Solution. Consider all the distances XY between two different drawn points. Since a finite number of points are drawn, there are only finitely many distances, and one of them, say d, is minimal. Let A and B be two of the drawn points such that $AB = d$. Now, by definition, A and B are not the same point. To verify part (b), let C be any drawn point other than A and B. Then $AC \geq d = AB$, so C is *not* closer to A than B is. The verification of part (c) is similar.

3. The number 2011 is written on a blackboard. It is permitted to transform the numbers on it by two types of moves:

   (a) Given a number n, we can erase n and write two nonnegative integers a and b such that $a + b = n$.

   (b) Given two numbers a and b, we can erase them and write their difference $a - b$, assuming this is positive or 0.

   Is it possible that after a sequence of such moves, there is only one number on the blackboard and it is 0?

   Solution. The answer is no.

   We claim that the sum of the numbers on the blackboard is always odd. Indeed, the initial sum, 2011, is odd, and when a move of type (a) is performed, the sum does not change. When a move of type (b) is performed, the sum decreases by

$$(a + b) - (a - b) = 2b,$$

   an *even* number. Since odd $-$ even $=$ odd, the sum will remain odd. Thus the desired final state, in which the sum (0) is even, is not achievable.

4. Let X, Y, and Z be points on one side of a line AB such that

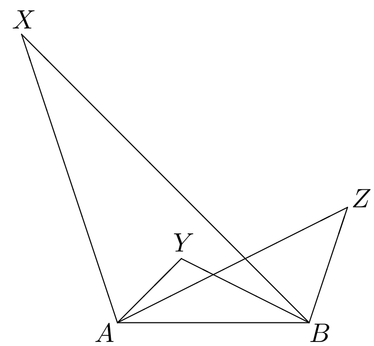
$$\triangle XAB \sim \triangle BYA \sim \triangle ABZ.$$

   Prove that $\triangle XYZ$ is similar to all these triangles.

Solution. We will prove that $\triangle XYZ \sim \triangle XAB$. Because $\angle AXB = \angle YXZ$ if and only if $\angle AXY = \angle BXZ$ and

$$\frac{ZX}{BX} = \frac{YX}{AX} \quad \text{if and only if} \quad \frac{ZX}{YX} = \frac{BX}{AX},$$

this is the same as proving $\triangle XAY \sim \triangle XBZ$. Note that



$$\angle YAX = \angle BAX - \angle BAY = \angle ZBA - \angle XBA = \angle ZBX.$$

Also note that

$$\frac{XA}{YA} = \frac{XA}{AB} \cdot \frac{AB}{YA} = \frac{AB}{BZ} \cdot \frac{BX}{AB} = \frac{BX}{BZ}.$$

Therefore $\triangle XAY \sim \triangle XBZ$ by SAS.

Remark. It is also easy to solve this problem by the method of complex numbers.

5. For positive integers n, define

$$a_n = 3^n + 6^n - 2^n.$$

Find, with proof, all primes that do not divide any of the numbers $a_1, a_2, a_3, \ldots$.

Solution. Answer: 2 and 3. It is clear that, for $n \geq 1$, the terms 6^n and 2^n are even while 3^n is odd, so $2 \nmid a_n$. Similarly, 3^n and 6^n are divisible by 3 but 2^n is not, so $3 \nmid a_n$.

Let $p \geq 5$ be a prime. We claim that $a_{p-2} \equiv 0 \bmod p$. We have

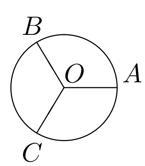
$$\begin{aligned} 6a_{p-2} &= 6(3^{p-2} + 6^{p-2} - 2^{p-2}) \\ &\equiv 2 \cdot 3^{p-1} + 6^{p-1} - 3 \cdot 2^{p-1} \\ &\equiv 2 + 1 - 3 \quad \text{(by Fermat's little theorem)} \\ &\equiv 0 \bmod p. \end{aligned}$$

Since p does not divide 6, p must divide a_{p-2}.

Remark. To motivate this solution, we note that $a_{-1} = 0$. To convert this illegal value $n = -1$ into a legal value, we add $p-1$ to n, which by Fermat's little theorem does not affect any of the terms mod p.

6. Let R be the region consisting of all points inside or on the boundary of a given circle of radius 1. Find, with proof, all positive real numbers d such that it is possible to color each point of R red, green or blue such that any two points of the same color are separated by a distance less than d.

Solution. The answer is all $d \geq \sqrt{3}$.



If $d \geq \sqrt{3}$, refer to the diagram at right. Color $120°$ sectors OAB, OBC, and OCA red, green, and blue respectively, including their boundaries on the circle. For the boundaries between the sectors, color OA red, OB green, and OC blue, making point O red. Then it is clear that the red sector can be covered by a circle of diameter $AB = \sqrt{3}$, so all points in it have a distance less than $\sqrt{3}$ except A and B, but B is not even red. So all red points have a distance less than $\sqrt{3}$, and a similar argument can be made for the green and the blue.

Now assume that $d < \sqrt{3}$. Let $\theta < 120°$ be the angle corresponding to a chord of length d. Let n be a positive integer large enough so that

$$\frac{1}{3n+1} < \frac{120° - \theta}{120°},$$

that is,

$$\theta < 120° - \frac{120°}{3n+1}.$$

Define points $P_0, P_1, P_2, \ldots$ recursively as follows: P_0 is any point on the circumference, and for $i \geq 0$, P_{i+1} is the counter-clockwise rotation of P_i by the angle

$$\alpha = 120° - \frac{120°}{3n+1} = 360° \cdot \frac{n}{3n+1}.$$

Note that

$$\theta < \alpha < 360° - \theta,$$

so for each $i \geq 0$, $P_i P_{i+1} > d$ and P_i and P_{i+1} are different colors. Also note that

$$\theta < 2\theta < 2\alpha < 240^\circ < 360^\circ - \theta,$$

so P_i and P_{i+2} are different colors. Thus P_i, P_{i+1}, and P_{i+2} are all different colors for each i. If we assume P_0 is red, P_1 is green, and P_2 is blue, we will get P_3 red, P_4 green, and so on until P_{3n+1} is green. But the total angle of rotation from P_0 to P_{3n+1} is

$$(3n+1)\alpha = (3n+1) \cdot 360^\circ \cdot \frac{n}{3n+1} = n \cdot 360^\circ,$$

so P_0 and P_{3n+1} are the same point and we have a contradiction.

7. Let $n > 1$ be an integer. Three complex numbers have the property that their sum is 0 and the sum of their nth powers is also 0. Prove that two of the three numbers have the same absolute value.

Solution. Given that

$$a + b + c = 0 \qquad \text{and} \qquad a^n + b^n + c^n = 0,$$

let

$$t = ab + bc + ca \qquad \text{and} \qquad u = abc.$$

Then a, b, and c are the roots of the polynomial

$$f(z) = (z-a)(z-b)(z-c) = z^3 + tz - u.$$

If $t = 0$, then a, b, and c are the three cube roots of u and thus all have the same absolute value; otherwise we can normalize, dividing a, b, and c by a square root of t, so that $t = 1$. Now we have

$$f(z) = (z-a)(z-b)(z-c) = z^3 + z - u.$$

Define $p_k = a^k + b^k + c^k$ for nonnegative integers k (here we make the convention that $0^0 = 1$). It is not hard to compute that $p_0 = 3$, $p_1 = 0$, $p_2 = -2$, and, for $k \geq 0$,

$$p_{k+3} = up_k - p_{k+1}. \tag{1}$$

This recursion allows us to think of each p_k as a polynomial (with integer coefficients) in u. Our plan will be to prove that, for $k > 1$, these polynomials have only real roots. Then, given that $p_n = 0$, we can deduce that $f(z)$ is a real polynomial and therefore either has all real roots (which is impossible, since the sum of the 2nd powers of the roots is -2) or has a pair of complex conjugate roots which have the same absolute value.

The leading term of p_k follows a pattern which is not difficult to verify by induction. It is:

$$\begin{array}{ll} 3u^i & \text{if } k = 3i,\ i \geq 0 \\ \frac{ki}{2}u^{i-1} & \text{if } k = 3i+1,\ i \geq 1 \\ -ku^i & \text{if } k = 3i+2,\ i \geq 0 \end{array}$$

In particular, for $k \neq 1$, the leading coefficients of p_k and p_{k+3} have the same sign, and the degree of p_{k+3} is one more than the degree of p_k. We will prove that the roots of p_k alternate with the roots of p_{k+3} on the number line, with no two coinciding, beginning and ending with a root of p_{k+3}. When p_k is constant (that is, for $k = 0$, 2, or 4) this statement is trivial, and we will use it as the base of an induction. For all other k, the induction hypothesis tells us there is exactly one root of p_{k-3} between each pair of roots of p_k, and it suffices to prove the following statement:

If $p_k = 0$, then p_{k+3} and p_{k-3} have opposite signs.

For brevity we will prove this only when $x = p_{k-3}$ is positive and u is also positive; the proof readily generalizes when one or both are negative. (By the induction hypothesis $p_{k-3} \neq 0$, and if $u = 0$ then (1) gives $p_{k+3} = -p_{k+1} = p_{k-1} = -p_{k-3}$ for $k \geq 4$.) Assume for contradiction that $y = p_{k+3} \geq 0$. Using (1) repeatedly, we get

$$p_{k+1} = up_k - p_{k+3} = -y$$
$$p_{k-2} = up_{k-3} - p_k = ux$$
$$p_{k-1} = up_{k-2} - p_{k+1} = u^2x + y.$$

Note that p_{k-1}, p_{k-2}, p_{k-3} are all positive. Then, using the reverse recursion

$$p_i = \frac{p_{i+1} + p_{i+3}}{u},$$

we get that $p_i > 0$ for all $i < k$. Since $p_2 = -2$, this is a contradiction.