INFINITE SUMS AND GENERATING FUNCTIONS

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1. Start with a triangle. Draw 3 lines connecting the midpoints of the sides and remove the middle triangle. Do the same for each of three remaining triangles, then do the same for each of nine remaining triangles . Repeat this process infinitely many times. What remains is called Sierpenski gasket. Find its area.

2. Starting with a vertical pile of identical books, shift the books (except the bottom one) to the right as far as possible. How far to the right can this "arch" go? Can one walk on such arch as on a bridge?

Two previous problems involve taking sum of infinitely many numbers. My first example of such sum is

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

3. Check if the following infinite sums are finite

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} + \cdots$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} + \cdots$$

4. The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots$$

(a) Show that

$$\zeta(s) = \left(\frac{1}{1 - \frac{1}{p_1^s}}\right)\left(\frac{1}{1 - \frac{1}{p_2^s}}\right)\cdots\left(\frac{1}{1 - \frac{1}{p_n^s}}\right)\cdots,$$

where the product in the right hand side is taken over all prime numbers $p_1, p_2, \ldots$.

(b) Use (a) to prove that there are infinitely many prime numbers.

(c) Show that

$$(\zeta(2) - 1) + (\zeta(3) - 1) + \cdots + (\zeta(n) - 1) + \cdots = 1$$

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The *generating function* for a sequence f_n is the expression

$$f(t) = f_0 + f_1 t + f_2 t^2 + \cdots + f_n t^n + \cdots$$

If the sequence is finite then $f(t)$ is a polynomial. In general, a generating function is "an infinite polynomial" which is usually called a *power series.*

5. Fix a positive integer k. Let $f_n = \binom{n}{k}$. Then $f(t) = (1+t)^k$.

6. Define operations of multiplication and addition for power series and check that they satisfy associativity, commutativity and distributivity laws. Use these operations to "check" the identity

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots .$$

When can one divide one power series by another one?

7. Find the generating functions for the following sequences

(a) $f_n = n$;

(b) $f_n = \binom{n+k-1}{k}$.

8. Fix a positive integer k. Show that every positive integer k can be uniquely written in the form

$$n = \binom{a_1}{1} + \binom{a_2}{2} + \cdots + \binom{a_k}{k}$$

for some $0 \leq a_1 \leq a_2 \leq \cdots \leq a_k$.

9. Let f_n be the Fibonacci sequence ($f_0 = 1, f_1 = 1, f_{n+2} = f_n + f_{n+1}$).

(a) Show that

$$f(t) = \frac{1}{1-t-t^2}.$$

(b) Let α and β be two roots of the equation $1 - t - t^2 = 0$. Show that

$$\frac{1}{1-t-t^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{t-\alpha} + \frac{1}{t-\beta} \right).$$

(c) Show that

$$f_n = \frac{(-1)^n}{\sqrt{5}} \left(\left(\frac{-1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{-1-\sqrt{5}}{2} \right)^{n+1} \right).$$

10. In the last century all bus tickets in Moscow had 6-digit numbers. It was believed that a ticket brings luck if the sum of the first three digits equals the sum of the last three digits.

(a) Show that the number of the lucky tickets equals to the constant term in the expression

$$(1 + t + \cdots + t^9)^3 (1 + t^{-1} + \cdots + t^{-9})^3.$$

(b) Show that the number of lucky tickets equals $\binom{32}{5} - 6\binom{22}{5} + 15\binom{12}{5}$.

(c) Estimate the probability to get a lucky ticket.

11. (Catalan numbers) Let C_n denote the number of triangulations of the $n+2$-gon by $n-1$ non-intersecting diagonals.

(a) Prove the relation

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0.$$

(b) Show that the generating function $C(t)$ satisfies the equation

$$tC^2(t) - C(t) + 1 = 0.$$

(c) Show that

$$C(t) = \frac{1 - \sqrt{1-4t}}{2t}$$

and

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

13. Let q_n be the number of ways to pay n cents using pennies, nickels, dimes and quarters.

(a) Show that

$$q(t) = \frac{1}{(1-t)(1-t^5)(1-t^{10})(1-t^{25})},$$

(b) In how many ways one can pay 1 dollar by pennies, nickels, dimes and quarters?

14. Let P_n be the number of ways to write n as a sum of positive integers. Show that

$$P(t) = \frac{1}{(1-t)(1-t^2)(1-t^3)\cdots}.$$

15. Prove the identity

$$(1+t)(1+t^2)(1+t^3)\cdots = \frac{1}{(1-t)(1-t^3)(1-t^5)\cdots}.$$

16. (Euler's identity) Let

$$Q(t) = 1 - (t + t^2) + (t^5 + t^7) + \cdots + (-1)^n (t^{\frac{3n^2-n}{2}} + t^{\frac{3n^2+n}{2}}) + \cdots$$

Show that $P(t)Q(t) = 1$.

17. Define the exponential power series by

$$e^t = 1 + t + \frac{t^2}{2!} + \cdots \frac{t^n}{n!} + \cdots.$$

Check the identity

$$e^t e^s = e^{t+s}.$$

18. John and Mary have to send 10 postcards for Christmas. Mary writes greetings and John writes addresses on the envelopes. Their daughter Jane puts the postcards to the envelopes randomly because she can not read yet. Estimate the probability that none of 10 people gets the right postcard.

19. Show that if f_n is periodic then the generating function is rational, i.e. $f(t) = \frac{p(t)}{q(t)}$ for some polynomials $p(t)$ and $q(t)$.

20. Show that $f(t)$ is rational if and only if there exist numbers $c_1, \ldots, c_k$ such that starting from some n

$$f_{n+k} = c_k f_n + \cdots + c_1 f_{n+k-1}.$$

21. Let f_n be the Fibonacci sequence. Find the generating function for f_n^2.

22. Show that $f(t)$ is rational if and only if there exist polynomials $p_1, \ldots, p_l$ and numbers $q_1, \ldots, q_l$ such that starting from some n

$$f_n = p_1(n)q_1^n + \cdots + p_l(n)q_l^n.$$

23. The Hadamard product of two power series $a(t)$ and $b(t)$ is defined by the formula

$$a \circ b(t) = a_0 b_0 + a_1 b_1 t + \cdots + a_n b_n t^n + \cdots$$

Show the Hadamard product of two rational functions is rational.

References

1. S. Lando, Lectures on generating functions.
2. R. Graham, D. E. Knuth, O. Patashnik, Concrete Mathematics.

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