Out of the shadows ...

Dave Auckly

1. Describe why the formulas for the area of a circle, volume of a pyramid, volume of a cone, volume of a sphere and surface area of a sphere are what they are. In particular:
   (a) How many congruent pyramids can you fit in a cube to fill it? What is the smallest number you can do this with?
   (b) How do length, area and volume scale under dilations? Why?
   (c) Can you relate the volume of a hemisphere, cone and cylinder?
   (d) Can you drill a hole in a sphere and lay it flat so that the process is obtained by revolving the "unrolling phone book" about an axis?

2. This problem leads to the space form law of sines.
   (a) What is the formula for the height of a triangle given two sides and angle between them?
   (b) What is the area of a triangle given two sides and angle between them?
   (c) If a shape having area A makes an angle θ with the ground, what is the area of its shadow given that the light is directly overhead and very far away?
   (d) Draw a spherical triangle on a sphere of radius R and label its sides a, b, and c. We will also let these letters represent the lengths of the sides. Label the angles opposite from these sides α, β, and γ respectively. We will also use these letters to represent the corresponding vertices. Let O denote the center of the sphere.
   (e) What is angle β, O, γ?
   (f) What is the area of triangle β, O, γ?
   (g) What is the area of the projection of triangle β, O, γ projected to the plane containing the altitude of the triangle through γ and O?
   (h) How would the area computed in part g) compare with the area of the projection of triangle α, O, γ?
   (i) Write the relation from part h) mathematically. This is the space form law of sines.
   (j) Use the approximation $\sin(x) \approx x$ for x close to zero to understand what happens to your equation as R grows.

3. Put three dots on a page near the vertices of an equilateral triangle. Now put between two and ten additional dots inside the triangle. Play the following game with a colleague: Take turns drawing curves connecting dots so that no two curves cross. The person to make the last move wins.

   (a) Decide who will win.

   (b) Make a table listing the total number of vertices, edges and faces at the end of each game played.

   (c) Write down any patterns that you see.

   (d) Prove your answers.

4. This problem leads to the classification of Platonic solids. In a platonic solid each face has the same number of sides, say s sides. In addition each vertex meets the same number of edges. This is called the degree of the vertex and we denote it by d.

   (a) Given that each face has s sides, write an equation for the number of faces as a function of the number of edges. Hint: count the number of "sides" of edges.

   (b) Given that each vertex has degree d, write a formula for the number of vertices in terms of the number of faces. Hint: count corners.

   (c) Write the number of vertices as a function of the number of edges.

   (d) Use Euler's formula $V - E + F = 2$ to derive an equation relating $1/d$, $1/s$ and $1/E$.

   (e) Find all natural number solutions to the equation in part (d).

   (f) Construct/draw a picture of a solid realizing the numbers for each solution.

5. This problem investigates the relationship between irrotational fields and gradient fields.

   (a) Show that $\operatorname{curl}(\operatorname{grad}(f)) = 0$ for any function f.

   (b) Let $H^1 := \{\mathbf{V} | \operatorname{curl}(\mathbf{V}) = 0\}/\{\operatorname{grad}(f)\}$ for vector fields defined on $\mathbb{R}^2 - \{(\pm 1, 0)\}$. Define $\Psi : H^1 \to \mathbb{R}^2$ by $\Psi(\mathbf{V} = (\int_{\gamma_-} \mathbf{V} \cdot (\mathbf{i}dx + \mathbf{j}dy), \int_{\gamma_+} \mathbf{V} \cdot (\mathbf{i}dx + \mathbf{j}dy))$ For simple closed curves $\gamma_\pm$ linking $(\pm 1, 0)$. Show that this is well-defined.

   (c) Show that Ψ is surjective. Hint: Consider $(2\pi((X-1)^2 + y^2))^{-1}((x-1)\mathbf{j} - y\mathbf{i})$.

   (d) Explain why it is plausible that $\Psi(\mathbf{V}) = 0$ implies that $\int_\gamma \mathbf{V} \cdot (\mathbf{i}dx + \mathbf{j}dy) = 0$ for all γ. Use this and a formula for $\int_\delta \operatorname{grad}(f) \cdot (\mathbf{i}dx + \mathbf{j}dy)$ vor curves δ to show that Ψ is injective.