Generating Functions

Claudiu Raicu

April 5, 2011

Definition 0.1. *Given a sequence of numbers* $(a_n)_{n\geq 0} = a_0, a_1, a_2, \cdots$, *the* generating function *of the sequence* $(a_n)_{n\geq 0}$ *is the* infinite formal sum *(or* power series*)*

$$a_0 + a_1X + a_2X^2 + \cdots = \sum_{n\geq 0} a_nX^n.$$

Conversely, to any power series we can associate the sequence consisting of its coefficients.

1 Warm-up

1. The generating function of the sequence $1, 1, 1, \cdots$ is
$$1 + X + X^2 + \cdots = \sum_{n\geq 0} X^n = \frac{1}{1-X}.$$
2. The generating function of the sequence $1, r, r^2, \cdots$ is
$$1 + rX + r^2X^2 + \cdots = \sum_{n\geq 0} r^nX^n = \frac{1}{1-rX}.$$
3. The generating function of the sequence $1, 0, 1, 0, 1, 0, \cdots$ is
$$\frac{1}{1-X^2}.$$
4. The generating function of the sequence $1, 2, 3, 4, \cdots$ is
$$\frac{1}{(1-X)^2}.$$

2 Recurrence relations

1. Determine the sequences (a_n) and (b_n) given by
$$a_0 = 0, \quad a_{n+1} = 2a_n + 1,$$
and
$$b_0 = 1, \quad b_{n+1} = 2b_n + 1.$$
2. Consider the Fibonacci sequence F_n, given by $F_0 = 0$, $F_1 = 1$, and
$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$
Determine F_n.

3 Binomial coefficients

1. (binomial expansion) For a positive integer n, we have

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k = \sum_{k\geq 0} \binom{n}{k} X^k.$$

2. Show that

$$\sum_{n\geq 0} \binom{n}{k} X^n = \frac{X^k}{(1-X)^{k+1}}.$$

Alternatively,

$$\sum_{n\geq 0} \binom{n+k}{k} X^n = \frac{1}{(1-X)^{k+1}}.$$

3. Show that for nonnegative integers n, m, r

$$\sum_{i} \binom{n}{i}\binom{m}{r-i} = \binom{n+m}{r},$$

4. (generalized binomial expansion) For a real number α, we define

$$(1+X)^\alpha := \sum_{k\geq 0} \binom{\alpha}{k} X^k.$$

Show that $(1+X)^\alpha \cdot (1+X)^\beta = (1+X)^{\alpha+\beta}$.

5. Show that

$$F_{n+1} = \sum_{k\geq 0} \binom{n-k}{k},$$

where F_n denotes the n-th Fibonacci number ($F_0 = 0$, $F_1 = 1$).

6. Show that

$$\sum_{k} \binom{n+k}{2k} 2^{n-k} = \frac{2^{2n+1}+1}{3}.$$

7. Show that

$$\sum_{k} \binom{m}{k}\binom{n+k}{m} = \sum_{k} \binom{m}{k}\binom{n}{k} 2^k.$$

4 Catalan numbers

1. Let C_n denote the number of different ways $n+1$ factors can be completely parenthesized. For $n = 3$ we have the following five different parenthesizations of four factors:

$$((xy)z)t,\ (x(yz))t,\ (xy)(zt),\ x((yz)t),\ x(y(zt)).$$

Determine the sequence C_n. The numbers C_n are called Catalan numbers.

2. For $m, n \geq 0$, show that

$$\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \binom{n-1}{m-1}.$$

3. Show that

$$\sum_k \binom{2k}{k}\binom{n}{k}(-1)^k 2^{-k} = \begin{cases} \binom{n}{n/2}2^{-n} & n \text{ even,} \\ 0 & n \text{ odd} \end{cases}.$$

5 Divisibility

1. For which values of n are all the binomial coefficients

$$\binom{n}{1}, \binom{n}{2}, \cdots \binom{n}{n-2}, \binom{n}{n-1}$$

even?

2. For which values of n are all the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \cdots \binom{n}{n-1}, \binom{n}{n}$$

odd?

3. Show that for any n, the number of odd binomial coefficients in the sequence

$$\binom{n}{0}, \binom{n}{1}, \cdots \binom{n}{n-1}, \binom{n}{n}$$

is a power of 2.

4. More generally, show that if $n, p \geq 2$ are integers, then the number of elements of the set

$$A = \{(k_1, \cdots, k_p) : k_1 + \cdots + k_p = n,\ k_i \geq 0,\ \text{and } \frac{n!}{k_1! \cdots k_p!} \text{ is odd}\}$$

is a power of p.